

# On the homology of almost Calabi-Yau algebras associated to $SU(3)$ modular invariants

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## Abstract

We compute the Hochschild homology and cohomology, and cyclic homology, of almost Calabi-Yau algebras for  $SU(3)$   $ADE$  graphs. These almost Calabi-Yau algebras are a higher rank analogue of the pre-projective algebras for Dynkin diagrams, which are  $SU(2)$ -related constructions. The Hochschild (co)homology and cyclic homology of  $A$  can be regarded as invariants for the braided subfactors associated to the  $SU(3)$  modular invariants.

## 1 Introduction

The classical McKay correspondence appears in various contexts. It relates finite subgroups  $\Gamma$  of  $SU(2)$  with the algebraic geometry of the quotient Kleinian singularities  $\mathbb{C}^2/\Gamma$  [33] but also with the classification of  $SU(2)$  modular invariants and quantum subgroups of  $SU(2)$  [8, 40, 31, 32, 37, 1, 2, 4, 5].

Minimal resolutions of Kleinian singularities can be described via the moduli space of representations of the preprojective algebra associated to the action of  $\Gamma$  [12]. Preprojective algebras associated to graphs were introduced in [25], and it was shown that they are finite dimensional if and only if the graphs are of  $ADE$  type, that is, one of the simply laced Dynkin diagrams. The preprojective algebra  $A$  for an  $ADE$  Dynkin diagram is a Frobenius algebra, that is, there is a linear function  $f : A \rightarrow \mathbb{C}$  such that  $(x, y) := f(xy)$  is a non-degenerate bilinear form (this is equivalent to the statement that  $A$  is isomorphic to its dual  $\hat{A} = \text{Hom}(A, \mathbb{C})$  as left (or right)  $A$ -modules). There is an automorphism  $\beta$  of  $A$ , called the Nakayama automorphism of  $A$  (associated to  $f$ ), such that  $(x, y) = (y, \beta(x))$ , which yields an  $A$ - $A$  bimodule isomorphism  $\hat{A} \rightarrow {}_1A_\beta$  [39]. The Nakayama automorphism for each  $ADE$  graph was determined in [13, 14] (see also [7]). The preprojective algebra  $A$

has a finite resolution as an  $A$ - $A$  bimodule, which was used by Erdmann and Snashall to determine the Hochschild cohomology  $HH^\bullet(A)$  of  $A$  for the  $ADE$  graphs  $A_n$ , along with its ring structure [13], and  $HH^2(A)$  for the graphs  $D_n$  [14]. This finite resolution yields a projective resolution of  $A$  as an  $A$ - $A$  bimodule, which was used by Etingof and Eu to determine the Hochschild homology and cohomology, and cyclic homology, of  $A$  for all  $ADE$  Dynkin diagrams [15], along with the ring structure of the Hochschild cohomology [17]. The Hochschild homology and cohomology, cyclic homology, and ring structure of the Hochschild cohomology, for the preprojective algebra for the tadpole graphs  $T_n$  were obtained in [18]. The Hochschild homology and cohomology for the case of the affine Dynkin diagrams, which are the McKay graphs for the finite subgroups of  $SU(2)$ , were determined in [11].

More generally, one tries to understand singularities via a noncommutative algebra  $A$ , often called a noncommutative resolution, whose centre corresponds to the coordinate ring of the singularity [35]. The algebra should be finitely generated over its centre, and the desired favourable resolution is the moduli space of representations of  $A$ , whose category of finitely generated modules is derived equivalent to the category of coherent sheaves of the resolution. In the case of a quotient singularity  $\mathbb{C}^3/\Gamma$  for a finite subgroup  $\Gamma$  of  $SU(3)$ , the corresponding noncommutative algebra  $A$  is a Calabi-Yau algebra of dimension 3.

Calabi-Yau algebras arise naturally in the study of Calabi-Yau manifolds, providing a noncommutative version of conventional Calabi-Yau geometry. An algebra  $A$  is Calabi-Yau of dimension  $n$  if the bounded derived category of the abelian category of finite dimensional  $A$ -modules is a Calabi-Yau category of dimension  $n$ . In this case the global dimension of  $A$  is  $n$  [6]. The derived category of coherent sheaves over an  $n$ -dimensional Calabi-Yau manifold is a Calabi-Yau category of dimension  $n$  and they appear naturally in the study of boundary conditions of the  $B$ -model in superstring theory over the manifold. For more on Calabi-Yau algebras, see e.g. [6, 26].

In [26, Remark 4.5.7] Ginzburg introduced, in his terminology,  $q$ -deformed Calabi-Yau algebras. In the case where  $q$  is not a root of unity, these algebras are Calabi-Yau algebras of dimension 3. We study these algebras in the case where  $q$  is a root of unity, which are the  $SU(3)$  generalizations of preprojective algebras for the Coxeter-Dynkin diagrams  $ADE$ . We call these algebras *almost Calabi-Yau algebras*. In a recent work [23], we determined the Nakayama automorphism for each  $\mathcal{ADE}$  graph, and constructed a finite resolution of  $A$  as an  $A$ - $A$  bimodule, see (5).

Our interest in these almost Calabi-Yau algebras came from subfactor theory, and in particular, braided subfactors of von Neumann algebras, which provide a framework for studying two dimensional conformal field theories and their modular invariant partition functions. In the case of Wess-Zumino-Witten models associated to  $SU(n)$  at level  $k$ , the Verlinde algebra is a non-degenerately braided system of endomorphisms  ${}_N\mathcal{X}_N$ , labelled by the positive energy representations of the loop group of  $SU(n)_k$  on a type  $III_1$  factor  $N$ , with fusion rules  $\lambda\mu = \bigoplus_\nu N_{\lambda\nu}^\mu \nu$  which exactly match those of the positive energy representations [36]. The fusion matrices  $N_\lambda = [N_{\rho\lambda}^\sigma]_{\rho,\sigma}$  are a family of commuting normal matrices which give a representation themselves of the fusion rules of the positive energy representations of the loop group of  $SU(n)_k$ ,  $N_\lambda N_\mu = \sum_\nu N_{\lambda\nu}^\mu N_\nu$ . This family  $\{N_\lambda\}$  of

fusion matrices can be simultaneously diagonalised:

$$N_\lambda = \sum_\sigma \frac{S_{\sigma,\lambda}}{S_{\sigma,0}} S_\sigma S_\sigma^*,$$

where 0 is the trivial representation, and the eigenvalues  $S_{\sigma,\lambda}/S_{\sigma,0}$  and eigenvectors  $S_\sigma = [S_{\sigma,\mu}]_\mu$  are described by the statistics  $S$  matrix. The key structure in the conformal field theory is the modular invariant partition function  $Z$ . In the subfactor setting this is realised by a braided subfactor  $N \subset M$  where trivial (or permutation) invariants in the ambient factor  $M$  when restricted to  $N$  yield  $Z$ . This would mean that the dual canonical endomorphism is in  $\Sigma({}_N\mathcal{X}_N)$ , i.e. decomposes as a finite linear combination of endomorphisms in  ${}_N\mathcal{X}_N$ . Indeed if this is the case for the inclusion  $N \subset M$ , then the process of  $\alpha$ -induction allows us to analyse the modular invariant, providing two extensions of  $\lambda$  on  $N$  to endomorphisms  $\alpha_\lambda^\pm$  of  $M$ , such that the matrix  $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$  is a modular invariant [4, 3, 20]. The action of the system  ${}_N\mathcal{X}_N$  on the  $N$ - $M$  sectors  ${}_N\mathcal{X}_M$  produces a *nimrep* (non-negative matrix integer representation of the fusion rules)  $G_\lambda G_\mu = \sum_\nu N_{\lambda\nu}^\mu G_\nu$ , whose spectrum reproduces exactly the diagonal part of the modular invariant, i.e.

$$G_\lambda = \sum_\sigma \frac{S_{\sigma,\lambda}}{S_{\sigma,0}} \psi_\sigma \psi_\sigma^*,$$

with the spectrum of  $G_\lambda$  given by  $G_\lambda = \{S_{\mu,\lambda}/S_{\mu,0}$  with multiplicity  $Z_{\mu,\mu}\}$  [5, Theorem 4.16].

The systems  ${}_N\mathcal{X}_N$ ,  ${}_N\mathcal{X}_M$ ,  ${}_M\mathcal{X}_M$  are (the irreducible objects of) tensor categories of endomorphisms with the Hom-spaces as their morphisms. Thus  ${}_N\mathcal{X}_N$  gives a braided modular tensor category, and  ${}_N\mathcal{X}_M$  a module category.

In our work we have focused on braided subfactors associated to  $SU(3)$  modular invariants, which are labeled by a family of graphs which we call the  $SU(3)$   $\mathcal{ADE}$  graphs. The complete list of the  $SU(3)$   $\mathcal{ADE}$  graphs are illustrated in [24, Figures 5-9]. For positive integer  $k < \infty$  we have a braided modular tensor category  ${}_N\mathcal{X}_N = \{\lambda_{(p,l)} \mid 0 \leq p, l, p+l \leq k\}$ , a non-degenerately braided system of endomorphisms on a type III<sub>1</sub> factor  $N$ , which is generated by  $\rho = \lambda_{(1,0)}$  and its conjugate  $\bar{\rho} = \lambda_{(0,1)}$ , where the irreducible endomorphisms  $\lambda_{(p,l)}$  satisfy the fusion rules of  $SU(3)_k$ :

$$\lambda_{(p,l)} \otimes \rho \cong \lambda_{(p,l-1)} \oplus \lambda_{(p-1,l+1)} \oplus \lambda_{(p+1,l)}, \quad \lambda_{(p,l)} \otimes \bar{\rho} \cong \lambda_{(p-1,l)} \oplus \lambda_{(p+1,l-1)} \oplus \lambda_{(p,l+1)}, \quad (1)$$

where  $\lambda_{(p',l')}$  is understood to be zero if  $p' < 0$ ,  $l' < 0$  or  $p' + l' \geq k + 1$ . Then a pair  $(\mathcal{G}, W)$ , of a cell system  $W$  (see Section 2) on an  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$  with Coxeter number  $k + 3$  yields a braided subfactor  $N \subset M$  and a module category  ${}_N\mathcal{X}_M$ , where the associated modular invariant, labeled by  $\mathcal{G}$ , is at level  $k$ . For such a braided subfactor, the almost Calabi-Yau algebra can be constructed via a monoidal functor  $F$ , which is essentially the module category  ${}_N\mathcal{X}_M$ , from the  $A_2$ -Temperley-Lieb category to the category  $\text{Fun}({}_N\mathcal{X}_M, {}_N\mathcal{X}_M)$  of additive functors from  ${}_N\mathcal{X}_M$  to itself.

The  $A_2$ -Temperley-Lieb category constructed in [23] used ideas from planar algebras, and in particular, the  $A_2$ -planar algebras of [22] (see also an earlier construction of the  $A_2$ -Temperley-Lieb category in [10], and of the Temperley-Lieb category in [34, 38]).

For  $m_i, n_i \geq 0$ , an  $A_2$ -( $m_2, n_2$ ), ( $m_1, n_1$ )-tangle  $T$  is a tangle on an rectangle with  $m_2 + n_2$ ,  $m_1 + n_1$  vertices along the top, bottom edges respectively, generated by  $A_2$



Figure 1:  $A_2$  webs

webs (see Figure 1) such that every free end of  $T$  is attached to a vertex along the top or bottom of the rectangle in a way that respects the orientation of the strings, every vertex has a string attached to it, and the tangle contains no elliptic faces. We call a vertex a source vertex if the string attached to it has orientation away from the vertex. Similarly, a sink vertex will be a vertex where the string attached has orientation towards the vertex. Along the top, bottom edge the first  $m_i$  vertices are source, sink vertices respectively, and the last  $n_i$  are sink, source vertices respectively. Let  $V_{(m_2, n_2), (m_1, n_1)}^{A_2}$  be the quotient of the free vector space over  $\mathbb{C}$  with basis the  $A_2$ -( $m_2, n_2$ ), ( $m_1, n_1$ )-tangles, by the Kuperberg ideal generated by the Kuperberg relations K1-K3 [28]. Then at level  $k$ , the  $A_2$ -Temperley-Lieb category is defined to be a quotient of the category  $A_2-TL = \text{Mat}(C^{A_2})$  by the negligible morphisms, where  $C^{A_2}$  is the tensor category whose objects are projections in  $V_{(m, n), (m, n)}^{A_2}$  and whose morphisms are  $\text{Hom}(p_1, p_2) = p_2 V_{(m_2, n_2), (m_1, n_1)}^{A_2} p_1$ , for projections  $p_i \in V_{(m_i, n_i), (m_i, n_i)}^{A_2}$ ,  $i = 1, 2$ . We write  $A_2-TL_{(m, n)} = V_{(m, n), (m, n)}^{A_2}$ , and  $\rho, \bar{\rho}$  for the identity projections in  $A_2-TL_{(1, 0)}$ ,  $A_2-TL_{(0, 1)}$  respectively consisting of a single string with orientation downwards, upwards respectively. Then the identity diagram in  $A_2-TL_{(m, n)}$ , given by  $m + n$  vertical strings where the first  $m$  strings have downwards orientation and the next  $n$  have upwards orientation, is expressed as  $\rho^m \bar{\rho}^n$ . It is a linear combination of simple projections  $f_{(i, j)}$  for  $i, j \geq 0$ ,  $0 \leq i + j < m + n$  such that  $i - j \cong m - n \pmod{3}$ , and a simple projection  $f_{(m, n)}$ , where  $f_{(1, 0)} = \rho$ ,  $f_{(0, 1)} = \bar{\rho}$  and  $f_{(0, 0)}$  is the empty diagram. The morphisms  $f_{(p, l)} = \text{id}_{f_{(p, l)}}$  are generalized Jones-Wenzl projections. The  $f_{(p, l)}$  satisfy the fusion rules for  $SU(3)$  [23]:

$$f_{(p, l)} \otimes \rho \cong f_{(p, l-1)} \oplus f_{(p-1, l+1)} \oplus f_{(p+1, l)}, \quad f_{(p, l)} \otimes \bar{\rho} \cong f_{(p-1, l)} \oplus f_{(p+1, l-1)} \oplus f_{(p, l+1)}. \quad (2)$$

At level  $k$ , the negligible morphisms are the ideal  $\langle f_{(p, l)} | p + l = k + 1 \rangle$  generated by  $f_{(p, l)}$  such that  $p + l = k + 1$ . The  $A_2$ -Temperley-Lieb category is the quotient  $A_2-TL^{(k)} := A_2-TL / \langle f_{(p, l)} | p + l = k + 1 \rangle$ , which is semisimple with simple objects  $f_{(p, l)}$ ,  $p, l \geq 0$  such that  $p + l \leq k$  which satisfy the fusion rules (1) of  $SU(3)_k$ , that is we have (2) where  $f_{(p', l')}$  is understood to be zero if  $p' < 0$ ,  $l' < 0$  or  $p' + l' \geq k + 1$ . The  $A_2$ -Temperley-Lieb category  $A_2-TL^{(k)}$  may be identified with the braided modular tensor category  ${}_N\mathcal{X}_N$ , where the object  $f_{(p, l)} \in A_2-TL^{(k)}$  is identified with  $\lambda_{(p, l)} \in {}_N\mathcal{X}_N$ .

Then the monoidal functor  $F$  is given on the simple objects  $f_{(p, l)}$  of  $A_2-TL^{(k)}$  by

$$F(f_{(p, l)}) = \bigoplus_{i, j \in \mathcal{G}_0} G_{\lambda_{(p, l)}}(i, j) \mathbb{C}_{i, j}, \quad (3)$$

where  $\mathbb{C}_{i, j}$  are 1-dimensional  $R$ - $R$  bimodules, where  $R = (\mathcal{CG})_0$ . The category of  $R$ - $R$  bimodules has a natural monoidal structure given by  $\otimes_R$ . The functor  $F$  is defined on the morphisms of  $A_2-TL^{(k)}$  using the cell system  $W$  and the Perron-Frobenius eigenvector of  $\mathcal{G}$ , see [23, Section 2.9].

If  $\mathcal{G}^{\text{op}}$  denotes the opposite graph of  $\mathcal{G}$  obtained by reversing the orientation of every edge of  $\mathcal{G}$ , we have that  $F(\rho^m \bar{\rho}^n)$  is the  $R$ - $R$  bimodule with basis given by all paths of length  $m+n$  on  $\mathcal{G}$ ,  $\mathcal{G}^{\text{op}}$ , where the first  $m$  edges are on  $\mathcal{G}$  and the last  $n$  edges are on  $\mathcal{G}^{\text{op}}$ . In particular  $F(\rho^m) = (\mathbb{C}\mathcal{G})_m$ , so that we have the graded algebra  $\bigoplus_m F(\rho^m) = (\mathbb{C}\mathcal{G})$ , the path algebra of  $\mathcal{G}$ . The endomorphisms  $\rho^m$  are not irreducible however, but decompose into direct sums of the generalized Jones-Wenzl projections  $f_{(p,0)}$ . The natural algebra to consider is thus the graded algebra  $\Sigma = \bigoplus_j F(f_{(j,0)})$ , where the  $p^{\text{th}}$  graded part is  $\Sigma_p = F(f_{(p,0)})$ . The multiplication  $\mu$  is defined by  $\mu_{p,l} = F(f_{(p+l,0)}) : \Sigma_p \otimes_R \Sigma_l \rightarrow \Sigma_{p+l}$ , where  $f_{(p,l)} = \text{id}_{f_{(p,l)}}$ . The graded algebra  $\Sigma$  is isomorphic to the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$  [10, 23].

In Section 2 we introduce the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$  for a pair  $(\mathcal{G}, W)$  of a cell system  $W$  on an  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$ . Then in Section 2.1 we determine a periodic projective resolution of  $A$  as an  $A$ - $A$  bimodule, starting from the finite resolution of  $A$  determined in [23, Theorem 5.1], which will be used to determine the Hochschild (co)homology and cyclic homology of  $A$  in Sections 3-4. In Section 3.1 we use the projective resolution determined in Section 2.1 to construct a Hochschild homology complex for  $A$ , and introduce the cyclic homology of  $A$  in Section 3.2. We then determine the Hochschild and cyclic homology of  $A$  in Sections 3.3-3.5 for the graphs  $\mathcal{A}^{(n)}$ ,  $n = 4, 5, 6, 7$ ,  $\mathcal{D}^{(3k+3)}$ ,  $k \geq 1$ ,  $\mathcal{A}^{(n)*}$ ,  $n \geq 5$ ,  $\mathcal{D}^{(3k)*}$ ,  $k \geq 2$ ,  $\mathcal{E}^{(8)}$  and  $\mathcal{E}^{(8)*}$ . Finally in Section 4 we construct a Hochschild cohomology complex for  $A$  and use this to determine the Hochschild cohomology of  $A$  in the cases listed above.

The Hochschild (co)homology and cyclic homology of  $A$  can be regarded as invariants for the braided subfactors associated to the  $SU(3)$  modular invariants. Beginning with a pair  $(\mathcal{G}, W)$  given by a cell system  $W$  on an  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$ , we construct a braided subfactor  $N \subset M$  which yields a nimrep which recovers the graph  $\mathcal{G}$  as described above. Then we can construct the algebra  $A(\mathcal{G}, W)$  whose Hochschild (co)homology and cyclic homology only depends on the original pair  $(\mathcal{G}, W)$ , or equivalently, on the braided subfactor  $N \subset M$ .

## 2 Almost Calabi-Yau algebras

Let  $\mathcal{G}$  be a finite directed graph, and denote by  $\mathcal{G}_n$  the set of all paths on  $\mathcal{G}$  of length  $n$ . The vertices of  $\mathcal{G}$  are the paths of length 0. If  $a \in \mathcal{G}_1$  is an edge on  $\mathcal{G}$ , we denote by  $\tilde{a} \in \mathcal{G}_1^{\text{op}}$  the corresponding edge with opposite orientation on  $\mathcal{G}^{\text{op}}$ . The path algebra  $\mathbb{C}\mathcal{G} = \bigoplus_{k=0}^{\infty} (\mathbb{C}\mathcal{G})_k$  is the graded complex vector space with basis of the  $k^{\text{th}}$ -graded part  $(\mathbb{C}\mathcal{G})_k$  given by  $\mathcal{G}_k$ , where paths may begin at any vertex of  $\mathcal{G}$ . Multiplication of two paths  $a \in (\mathbb{C}\mathcal{G})_k$  and  $b \in (\mathbb{C}\mathcal{G})_l$  is given by concatenation of paths  $a \cdot b \in (\mathbb{C}\mathcal{G})_{k+l}$  (or simply  $ab$ ), with  $ab$  defined to be zero if  $r(a) \neq s(b)$ , where  $s(a)$ ,  $r(a)$  denotes the source, range vertex respectively of the path  $a$ . The commutator quotient  $\mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$  may be identified, up to cyclic permutation of the arrows, with the vector space spanned by cyclic paths in  $\mathcal{G}$ . Let  $\partial_a : \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}] \rightarrow \mathbb{C}\mathcal{G}$  be the derivation given by  $\partial_a(a_1 \cdots a_n) = \sum_j a_{j+1} \cdots a_n a_1 \cdots a_{j-1}$ , where the summation is over all indices  $j$  such that  $a_j = a$ . Then for a potential  $\Phi \in \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}]$ , which is some linear combination of cyclic paths in  $\mathcal{G}$ , we define the algebra  $A(\mathbb{C}\mathcal{G}, \Phi) = \mathbb{C}\mathcal{G}/\{\partial_a \Phi\}$ , which is the quotient of the path algebra by the two-sided ideal generated by the relations  $\partial_a \Phi \in \mathbb{C}\mathcal{G}$ , for all edges  $a$  of  $\mathcal{G}$ . The Hilbert

$$W\left(\begin{array}{c} b \nearrow c \\ \triangleleft a \end{array}\right) = \begin{array}{c} b \nearrow c \\ \triangleleft a \end{array} \quad \overline{W\left(\begin{array}{c} b \nearrow c \\ \triangleleft a \end{array}\right)} = W\left(\begin{array}{c} c \nearrow b \\ \triangleleft a \end{array}\right) = \begin{array}{c} c \nearrow b \\ \triangleleft a \end{array}$$

Figure 2: Cells associated to trivalent vertices

series  $H_A$  for  $A(\mathbb{C}\mathcal{G}, \Phi)$  is defined as  $H_A(t) = \sum_{k=0}^{\infty} H_{ji}^k t^k$ , where the  $H_{ji}^k$  are matrices which count the dimension of the subspace  $\{xj | x \in A(\mathbb{C}\mathcal{G}, \Phi)_k\}$ , where  $A(\mathbb{C}\mathcal{G}, \Phi)_k$  is the subspace of  $A(\mathbb{C}\mathcal{G}, \Phi)$  of all paths of length  $k$ , and  $i, j \in A(\mathbb{C}\mathcal{G}, \Phi)_0$ .

Ocneanu [31] defined a cell system  $W$  on any  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$ , associating a complex number  $W\left(\Delta_{i,j,k}^{(a,b,c)}\right)$ , now called an Ocneanu cell, to each closed loop of length three  $\Delta_{i,j,k}^{(a,b,c)}$  in  $\mathcal{G}$  as in Figure 2, where  $a, b, c$  are edges on  $\mathcal{G}$ , and  $i, j, k$  are the vertices on  $\mathcal{G}$  given by  $i = s(a) = r(c)$ ,  $j = s(b) = r(a)$ ,  $k = s(c) = r(b)$ . These cells satisfy two properties, called Ocneanu's type I, II equations respectively, which are obtained by evaluating the Kuperberg relations K2, K3 for an  $A_2$ -spider [28] using the identification in Figure 2:

(i) for any type I frame  $i \xleftarrow{a'} \xrightarrow{a} j$  in  $\mathcal{G}$  we have

$$\sum_{k, b_1, b_2} W\left(\Delta_{i,j,k}^{(a, b_1, b_2)}\right) \overline{W\left(\Delta_{i,j,k}^{(a', b_1, b_2)}\right)} = \delta_{a, a'} [2]_q \phi_i \phi_j$$

(ii) for any type II frame  $i_4 \xleftarrow{a_1} \xrightarrow{a_2} i_2$  in  $\mathcal{G}$  we have

$$\begin{aligned} \sum_{k, b_j} \phi_k^{-1} W\left(\Delta_{i_2, i_1, k}^{(a_2, b_1, b_2)}\right) \overline{W\left(\Delta_{i_2, i_3, k}^{(a_3, b_3, b_2)}\right)} W\left(\Delta_{i_4, i_3, k}^{(a_4, b_3, b_4)}\right) \overline{W\left(\Delta_{i_4, i_1, k}^{(a_1, b_4, b_1)}\right)} \\ = \delta_{a_1, a_4} \delta_{a_2, a_3} \phi_{i_4} \phi_{i_1} \phi_{i_2} + \delta_{a_1, a_2} \delta_{a_3, a_4} \phi_{i_1} \phi_{i_2} \phi_{i_3} \end{aligned}$$

Here  $(\phi_v)_v$  is the Perron-Frobenius eigenvector for the Perron-Frobenius eigenvalue  $\alpha = [3]_q$  of  $\mathcal{G}$ . The existence of these cells for the finite  $\mathcal{ADE}$  graphs was claimed by Ocneanu [31], and shown in [21] with the exception of the graph  $\mathcal{E}_4^{(12)}$ . These cells define a unitary connection on the graph  $\mathcal{G}$  which satisfy the Yang-Baxter equation [21, Lemma 3.2].

Two cell systems  $W_1, W_2$  on an  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$  are equivalent if, for each pair of adjacent vertices  $i, j$  of  $\mathcal{G}$ , we can find a family of unitary matrices  $(u(a, b))_{a, b}$ , where  $a, b$  are any pair of edges from  $i$  to  $j$ , such that

$$W_1(\Delta_{i_1, i_2, i_3}^{(a_1, a_2, a_3)}) = \sum_{a'_1, a'_2, a'_3} u(a_1, a'_1) u(a_2, a'_2) u(a_3, a'_3) W_2(\Delta_{i_1, i_2, i_3}^{(a'_1, a'_2, a'_3)}),$$

where  $a_l$  are edges from  $i_l$  to  $i_{l+1}$ , and the sum is over all edges  $a'_l$  from  $i_l$  to  $i_{l+1}$ ,  $l = 1, 2, 3$ .

There is up to equivalence precisely one connection on the graphs  $\mathcal{A}^{(m)}$ ,  $\mathcal{A}^{(2m+1)*}$ ,  $\mathcal{E}^{(8)}$ ,  $\mathcal{E}^{(8)*}$ ,  $\mathcal{E}_5^{(12)}$  and  $\mathcal{E}^{(24)}$ . For the graphs  $\mathcal{A}^{(2m)*}$  and  $\mathcal{E}_2^{(12)}$  there are precisely two inequivalent connections, which are obtained from each other by a  $\mathbb{Z}_2$  symmetry of the graph. This  $\mathbb{Z}_2$  symmetry is the conjugation of the graph in the case of  $\mathcal{E}_2^{(12)}$ . There is at least one connection for each graph  $\mathcal{D}^{(m)}$ ,  $m \not\equiv 0 \pmod{3}$ , and at least two inequivalent connections for each graph  $\mathcal{D}^{(3p)}$ , which are the complex conjugates of each other. There is at least



one connection for each graph  $\mathcal{D}^{(2m+1)*}$ , and at least two inequivalent connections for each graph  $\mathcal{D}^{(2m)*}$ , which are obtained from each other by a  $\mathbb{Z}_2$  symmetry of the graph. There are also at least two inequivalent connections for the graph  $\mathcal{E}_1^{(12)}$ , which are obtained from each other by conjugation of the graph.

For the  $SU(3)$   $\mathcal{ADE}$  graphs, we define the almost Calabi-Yau algebra  $A(\mathcal{G}, W)$  to be the graded quotient algebra

$$A(\mathcal{G}, W) := A(\mathbb{C}\mathcal{G}, \Phi_W),$$

where the potential  $\Phi_W$  is given by [23, equation (40)] (see also [26, Remark 4.5.7]):

$$\Phi_W = \sum_{abc} W(\Delta_{abc}) \Delta_{abc} \in \mathbb{C}\mathcal{G}/[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}],$$

where the summation is over all closed paths  $abc$  of length 3 on  $\mathcal{G}$ . The grading on  $\mathbb{C}\mathcal{G}$  descends to the quotient algebra  $A = A(\mathcal{G}, W)$ . These almost Calabi-Yau algebras were studied in [23] for all the cell systems constructed in [21]. Equivalent cell systems yield isomorphic almost Calabi-Yau algebras. For any cell system  $W$ , we can take its complex conjugate  $\overline{W}$  to obtain another (possibly equivalent) cell system. The almost Calabi-Yau algebra for  $\overline{W}$  is isomorphic to that for  $W$ . The conjugation  $\tau : {}_N\mathcal{X}_N \rightarrow {}_N\mathcal{X}_N$  on the braided system of endomorphisms of  $SU(3)_k$  on a factor  $N$ , given by the conjugation on the representations of  $SU(3)$ , induces a conjugation  $\tau : {}_N\mathcal{X}_M \rightarrow {}_N\mathcal{X}_M$  such that  $G_{\overline{\lambda}} = \tau G_{\lambda} \tau$ , where  $G_{\lambda} a = \lambda a$  for  $\lambda \in {}_N\mathcal{X}_N$ ,  $a \in {}_N\mathcal{X}_M$ . For any cell system  $W = W^+$ , this conjugation of the graph yields a conjugate cell system  $W^-$ , which might be equivalent to  $W^+$ . The almost Calabi-Yau algebra for  $W^-$  is anti-isomorphic to that for  $W^+$ .

The Hilbert series  $H_A(t)$  of  $A(\mathcal{G}, W)$ , for an  $SU(3)$   $\mathcal{ADE}$  graph  $\mathcal{G}$  with adjacency matrix  $\Delta_{\mathcal{G}}$ , Coxeter number  $h = k + 3$  and cell system  $W$ , is given by [23, Theorem 3.1]

$$H_A(t) = \frac{1 - Pt^h}{1 - \Delta_{\mathcal{G}}t + \Delta_{\mathcal{G}}^T t^2 - t^3}, \quad (4)$$

where  $P$  is the permutation matrix corresponding to a  $\mathbb{Z}_3$  symmetry of the graph. It is the identity for  $\mathcal{D}^{(n)}$ ,  $\mathcal{A}^{(n)*}$ ,  $n \geq 5$ ,  $\mathcal{E}^{(8)*}$ ,  $\mathcal{E}_l^{(12)}$ ,  $l = 1, 2, 4, 5$ , and  $\mathcal{E}^{(24)}$ . For the remaining graphs  $\mathcal{A}^{(n)}$ ,  $\mathcal{D}^{(n)*}$  and  $\mathcal{E}^{(8)}$ , let  $V$  be the permutation matrix corresponding to the clockwise rotation of the graph by  $2\pi/3$ . Then

$$P = \begin{cases} V^2 & \text{for } \mathcal{A}^{(n)}, n \geq 4, \\ V & \text{for } \mathcal{E}^{(8)}, \\ V^{2n} & \text{for } \mathcal{D}^{(n)*}, n \geq 5. \end{cases}$$

The numerator and denominator in (4) commute, since any permutation matrix which corresponds to a symmetry of the graph  $\mathcal{G}$  commutes with  $\Delta_{\mathcal{G}}$  and  $\Delta_{\mathcal{G}}^T$ .

## 2.1 Periodic resolution for almost Calabi-Yau algebras

We define a non-degenerate form on  $A$  by setting  $f$  to be the function which is 0 on every element of  $A$  of length  $< h - 3$ , and 1 on  $u_{i\nu(i)}$  for some  $i \in \mathcal{G}_1$ , where  $u_{j\nu(j)}$  denotes a generator of the one-dimensional top-degree space  $j \cdot A_{h-3} \cdot \nu(j)$ , where  $\nu$  is the

permutation of the vertices of  $\mathcal{G}$  given by the permutation matrix  $P$  in (4). Then using the relation  $(x, y) = (y, \beta(x))$  this determines the value of  $f$  on  $u_{j\nu(j)}$ , for all other  $j \in \mathcal{G}_1$ . We normalize the  $u_{j\nu(j)}$  such that  $f(u_{j\nu(j)}) = 1$  for all  $j \in \mathcal{G}_1$ . The image of the simple object  $f_{(k,0)} \in A_2\text{-TL}^{(k)}$  under the functor  $F$  given by (3) defines a unique permutation  $\nu$  of the graph  $\mathcal{G}$ , which is described as follows. The permutation  $\nu$  of the graph is given by the  $\mathbb{Z}_3$  symmetry which defines the permutation matrix  $P$  in (4) (note that there are no double edges on the graphs  $\mathcal{G}$  for which  $P$  is non-trivial). Then the Nakayama automorphism  $\beta$  of  $A$  is defined on  $\mathcal{G}$  by  $\beta = \nu$  [23, Theorem 4.6].

Now  $A$  has the following finite resolution as an  $A$ - $A$  bimodule [23, Theorem 5.1]:

$$0 \rightarrow \mathcal{N}[h] \xrightarrow{\iota_0} A \otimes_S A[3] \xrightarrow{\mu_3} A \otimes_S \tilde{V} \otimes_S A[1] \xrightarrow{\mu_2} A \otimes_S V \otimes_S A \xrightarrow{\mu_1} A \otimes_S A \xrightarrow{\mu_0} A \rightarrow 0. \quad (5)$$

Here  $S$  is the  $A$ - $A$  bimodule  $(\mathbb{C}\mathcal{G})_0$ , and  $V, \tilde{V}$  are the  $A$ - $A$  bimodules generated by  $\mathcal{G}_1, \mathcal{G}_1^{\text{op}}$  respectively. The  $A$ - $A$  bimodule  $\mathcal{N} = {}_1A_{\beta^{-1}}$  is equal to  $A$  as a vector space. The left  $A$ -action is given by concatenation, but the right  $A$ -action is twisted by the inverse of the Nakayama automorphism  $\beta$ , i.e.  $a \cdot x \cdot b = ax\beta^{-1}(b)$  for all  $a, b \in A, x \in \mathcal{N}$ . The connecting  $A$ - $A$  bimodule maps are given by

$$\mu_0(1 \otimes 1) = 1, \quad (6)$$

$$\mu_1(1 \otimes a \otimes 1) = a \otimes 1 - 1 \otimes a, \quad (7)$$

$$\mu_2(1 \otimes \tilde{a} \otimes 1) = \sum_{b, b' \in \mathcal{G}_1} W_{abb'}(b \otimes b' \otimes 1 + 1 \otimes b \otimes b'), \quad (8)$$

$$\mu_3(1 \otimes 1) = \sum_{a \in \mathcal{G}_1} a \otimes \tilde{a} \otimes 1 - \sum_{a \in \mathcal{G}_1} 1 \otimes \tilde{a} \otimes a, \quad (9)$$

$$\iota_0(1) = \sum_j w_j \otimes w_j^*,$$

where  $\{w_j\}$  is a homogeneous basis for  $A$ , and  $\{w_j^*\}$  is its corresponding dual basis, i.e.  $w_j w_j^* = u_{i\nu(i)}$  where  $i = s(w_j)$ . The  $A$ - $A$  bimodule  $B = B^{(1)} \otimes_S \cdots \otimes_S B^{(p)}$  is equipped with the *total grading* which comes from the grading on the graded  $A$ - $A$  bimodules  $B^{(i)}$ , that is,  $B = \bigoplus_{k=0}^{\infty} B_k$  where  $B_k = \bigoplus_{k_i: \sum_{i=1}^p k_i = k} B_{k_1}^{(1)} \otimes_S \cdots \otimes_S B_{k_p}^{(p)}$ .

For each  $SU(3)$   $\mathcal{ADE}$  graph, the Nakayama automorphism has order 3,  $\beta^3 = \text{id}$ , so we can make a canonical identification  $A = \mathcal{N} \otimes_A \mathcal{N} \otimes_A \mathcal{N}$ . We let  $\mathcal{N}^{(k)} := {}_1A_{\beta^{-k}}$ , for  $k \in \mathbb{Z}$ . In particular, we have  $A = \mathcal{N}^{(0)}$ ,  $\mathcal{N} = \mathcal{N}^{(1)}$  and  $\mathcal{N}^{(2)} = {}_1A_{\beta} = \mathcal{N} \otimes_A \mathcal{N}$ . Note that for graphs with trivial Nakayama automorphism,  $A = \mathcal{N}^{(k)}$  as  $A$ - $A$  bimodules, for all  $k \in \mathbb{Z}$ .

Applying the functor  $- \otimes_A \mathcal{N}$  to the exact sequence (5) we obtain the exact sequence:

$$0 \rightarrow \mathcal{N}^{(2)}[2h] \xrightarrow{\iota_2} A \otimes_S \mathcal{N}[h+3] \xrightarrow{\mu_7} A \otimes_S \tilde{V} \otimes_S \mathcal{N}[h+1] \xrightarrow{\mu_6} A \otimes_S V \otimes_S \mathcal{N}[h] \xrightarrow{\mu_5} A \otimes_S \mathcal{N}[h] \xrightarrow{\iota_1} \mathcal{N}[h] \rightarrow 0,$$

where  $\iota_1(x \otimes y) = xy$ ,  $\iota_2(a) = a \sum_j w_j \otimes w_j^*$ , where  $\{w_j\}$  is a homogeneous basis for  $A$  and  $\{w_j^*\}$  is its corresponding dual basis, and  $\mu_{i+4} = \mu_i$ . Similarly, applying the functor a second time we obtain the exact sequence:

$$0 \rightarrow A[3h] \xrightarrow{\iota_4} A \otimes_S \mathcal{N}^{(2)}[2h+3] \xrightarrow{\mu_{11}} A \otimes_S \tilde{V} \otimes_S \mathcal{N}^{(2)}[2h+1] \xrightarrow{\mu_{10}} A \otimes_S V \otimes_S \mathcal{N}^{(2)}[2h] \xrightarrow{\mu_9} A \otimes_S \mathcal{N}^{(2)}[2h] \xrightarrow{\iota_3} \mathcal{N}^{(2)}[2h] \rightarrow 0.$$



We now construct a projective resolution of  $A$ , that is, a resolution of  $A$  by projective modules. Setting  $\mu_4 = \iota_0 \iota_1$ ,  $\mu_8 = \iota_2 \iota_3$ , we obtain the following projective resolution of  $A$ , which is periodic with period 12:

$$\begin{aligned} \cdots \rightarrow A[3h] &\xrightarrow{\mu_{12}} A \otimes_S \mathcal{N}^{(2)}[2h+3] \xrightarrow{\mu_{11}} A \otimes_S \tilde{V} \otimes_S \mathcal{N}^{(2)}[2h+1] \xrightarrow{\mu_{10}} A \otimes_S V \otimes_S \mathcal{N}^{(2)}[2h] \\ &\xrightarrow{\mu_9} A \otimes_S \mathcal{N}^{(2)}[2h] \xrightarrow{\mu_8} A \otimes_S \mathcal{N}[h+3] \xrightarrow{\mu_7} A \otimes_S \tilde{V} \otimes_S \mathcal{N}[h+1] \xrightarrow{\mu_6} A \otimes_S V \otimes_S \mathcal{N}[h] \\ &\xrightarrow{\mu_5} A \otimes_S \mathcal{N}[h] \xrightarrow{\mu_4} A \otimes_S A[3] \xrightarrow{\mu_3} A \otimes_S \tilde{V} \otimes_S A[1] \xrightarrow{\mu_2} A \otimes_S V \otimes_S A \xrightarrow{\mu_1} A \otimes_S A \xrightarrow{\mu_0} A \rightarrow 0, \end{aligned} \quad (10)$$

where the connecting maps  $\mu_i$  are given by (6)-(9) for  $0 \leq i \leq 3$ ,  $\mu_4(x \otimes y) = xy \sum_j w_j \otimes w_j^*$ , where  $\{w_j\}$  is a homogeneous basis for  $A$  and  $\{w_j^*\}$  is its corresponding dual basis, and  $\mu_i = \mu_{i-4}$  for  $i \geq 5$ .

Thus we find that the Hochschild (co)homology of  $A$  is periodic with period 12, i.e. the grading is shifted by  $3h$  ( $-3h$ ) when the degree of the homology (respectively cohomology) is shifted by 12. In the case of trivial Nakayama automorphism the Hochschild (co)homology of  $A$  in fact has period 4.

### 3 The Hochschild homology of $A(\mathcal{G}, W)$

#### 3.1 The Hochschild homology complex

In this section we will construct a complex which determines the Hochschild homology of the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$ .

Let  $A^{\text{op}}$  denote the algebra with opposite multiplication, i.e.  $a \cdot b = ba$ , and define  $A^e = A^{\text{op}} \otimes_S A$ . Any  $A$ - $A$  bimodule becomes a left  $A^e$ -module, and vice versa, by defining the left action of  $A^e$  on  $A$  by  $(a \otimes b)x = bxa$  for all  $x \in A$ ,  $a \otimes b \in A^{\text{op}} \otimes_S A$ .

The Hochschild homology  $HH_\bullet(A)$  of  $A$  may be defined to be the derived functor  $HH_n(A) = \text{Tor}_n^{A^e}(A, A)$ , e.g. [29, Proposition 1.1.13], i.e. as the homology of the complex

$$\cdots \rightarrow P_2 \otimes_{A^e} A \rightarrow P_1 \otimes_{A^e} A \rightarrow P_0 \otimes_{A^e} A \rightarrow A \otimes_{A^e} A \rightarrow 0$$

where  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$  is any projective resolution of  $A$ .

For an  $A$ - $A$  bimodule  $M$ , denote by  $M^S$  the  $S$ -centralizer sub-bimodule given by all elements  $x \in M$  such that  $ix = xi$  for all  $i \in S$ . We make the following identifications, for  $k = 0, 1, 2$  (c.f. [15]):

$$\begin{aligned} (A \otimes_S \mathcal{N}^{(k)}) \otimes_{A^e} A &= (\mathcal{N}^{(k)})^S : & (x \otimes y) \otimes z &= y\beta^{-k}(zx), \\ (A \otimes_S V \otimes_S \mathcal{N}^{(k)}) \otimes_{A^e} A &= (V \otimes_S \mathcal{N}^{(k)})^S : & (x \otimes a \otimes y) \otimes z &= a \otimes y\beta^{-k}(zx), \\ (A \otimes_S \tilde{V} \otimes_S \mathcal{N}^{(k)}) \otimes_{A^e} A &= (\tilde{V} \otimes_S \mathcal{N}^{(k)})^S : & (x \otimes \tilde{a} \otimes y) \otimes z &= \tilde{a} \otimes y\beta^{-k}(zx), \end{aligned}$$

where the left and right hand sides have the same total degree. Thus, applying the functor  $-\otimes_{A^e} A$  to the resolution (10), we obtain the Hochschild homology complex:

$$\begin{aligned} \cdots \rightarrow A^S[3h] &\xrightarrow{\mu'_{12}} (\mathcal{N}^{(2)})^S[2h+3] \xrightarrow{\mu'_{11}} (\tilde{V} \otimes_S \mathcal{N}^{(2)})^S[2h+1] \xrightarrow{\mu'_{10}} (V \otimes_S \mathcal{N}^{(2)})^S[2h] \\ &\xrightarrow{\mu'_9} (\mathcal{N}^{(2)})^S[2h] \xrightarrow{\mu'_8} \mathcal{N}^S[h+3] \xrightarrow{\mu'_7} (\tilde{V} \otimes_S \mathcal{N})^S[h+1] \xrightarrow{\mu'_6} (V \otimes_S \mathcal{N})^S[h] \\ &\xrightarrow{\mu'_5} \mathcal{N}^S[h] \xrightarrow{\mu'_4} A^S[3] \xrightarrow{\mu'_3} (\tilde{V} \otimes_S A)^S[1] \xrightarrow{\mu'_2} (V \otimes_S A)^S \xrightarrow{\mu'_1} A^S \rightarrow 0, \end{aligned} \quad (11)$$

where the connecting maps are given, for  $k = 0, 1, 2, \dots$  by

$$\begin{aligned}
\mu'_{4k+1}(a \otimes x) &= \mu_{4k+1}(1 \otimes a \otimes 1) \otimes_{A^e} \beta^k(x) = (a \otimes 1 - 1 \otimes a) \otimes_{A^e} \beta^k(x) \\
&= x\beta^{-k}(a) - ax, \\
\mu'_{4k+2}(\tilde{a} \otimes x) &= \mu_{4k+2}(1 \otimes \tilde{a} \otimes 1) \otimes_{A^e} \beta^k(x) \\
&= \sum_{b, b' \in \mathcal{G}_1} W_{abb'}(b \otimes b' \otimes 1 + 1 \otimes b \otimes b') \otimes_{A^e} \beta^k(x) \\
&= \sum_{b, b' \in \mathcal{G}_1} W_{abb'}(b' \otimes x\beta^{-k}(b) + b \otimes b'x), \\
\mu'_{4k+3}(x) &= \mu_{4k+3}(1 \otimes 1) \otimes_{A^e} \beta^k(x) = \left( \sum_{a \in \mathcal{G}_1} a \otimes \tilde{a} \otimes 1 - 1 \otimes \tilde{a} \otimes a \right) \otimes_{A^e} \beta^k(x) \\
&= \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes (x\beta^{-k}(a) - ax), \\
\mu'_{4k+4}(y) &= \mu_{4k+4}(1 \otimes 1) \otimes_{A^e} \beta^{k+1}(y) = \left( \sum_j w_j \otimes w_j^* \right) \otimes_{A^e} \beta^{k+1}(y) \\
&= \sum_j w_j^* \beta(y) \beta^{-k}(w_j),
\end{aligned}$$

where  $a \in V$ ,  $x \in \mathcal{N}^{(k)}$ ,  $y \in \mathcal{N}^{(k+1)}$ ,  $\{w_j\}$  is a homogeneous basis for  $A$  and  $\{w_j^*\}$  is its corresponding dual basis.

We will now show that this complex has a self-duality. Using the non-degenerate form, we can make the identifications  $\mathcal{N}^{(k)} = (\mathcal{N}^{(2-k)})^*[h-3]$  by sending  $x \mapsto (-, x)$ . We can define a non-degenerate form on  $(V \oplus \tilde{V}) \otimes_S \mathcal{N}^{(k)}$  by  $(a_1 \otimes x_1, a_2 \otimes x_2) = \delta_{a_1, \beta^{k-1}(\tilde{a}_2)}(x_1, x_2)$  for  $x_1 \in \mathcal{N}^{(2-k)}$ ,  $x_2 \in \mathcal{N}^{(k)}$ , and  $a_1 \in V_1$ ,  $a_2 \in V_2$ , where  $V_i \in \{V, \tilde{V}\}$ ,  $i = 1, 2$ . For the  $\mathcal{A}^*$  graphs,  $V = \tilde{V}$  and we replace  $(V \oplus \tilde{V}) \otimes_S \mathcal{N}^{(k)}$  above by  $V \otimes_S \mathcal{N}^{(k)}$ . This allows us to make identifications  $V \otimes_S \mathcal{N}^{(k)} = (\tilde{V} \otimes_S \mathcal{N}^{(2-k)})^*[h-1]$ ,  $\tilde{V} \otimes_S \mathcal{N}^{(k)} = (V \otimes_S \mathcal{N}^{(2-k)})^*[h-1]$ , by sending  $a \otimes x \mapsto (-, a \otimes x)$ .

If we take the Hochschild homology sequence (11) and dualise, we get:

$$\begin{aligned}
&\dots \xleftarrow{(\mu'_{12})^*} A^S[-3h] \xleftarrow{(\mu'_{11})^*} (V \otimes_S A)^S[-3h] \xleftarrow{(\mu'_{10})^*} (\tilde{V} \otimes_S A)^S[-3h+1] \xleftarrow{(\mu'_9)^*} \\
&\quad \xleftarrow{(\mu'_9)^*} A^S[-3h+3] \xleftarrow{(\mu'_8)^*} \mathcal{N}^S[-2h] \xleftarrow{(\mu'_7)^*} (V \otimes_S \mathcal{N})^S[-2h] \xleftarrow{(\mu'_6)^*} \\
&\quad \xleftarrow{(\mu'_6)^*} (\tilde{V} \otimes_S \mathcal{N})^S[-2h+1] \xleftarrow{(\mu'_5)^*} \mathcal{N}^S[-2h+3] \xleftarrow{(\mu'_4)^*} (\mathcal{N}^{(2)})^S[-h] \xleftarrow{(\mu'_3)^*} \\
&\quad \xleftarrow{(\mu'_3)^*} (V \otimes_S \mathcal{N}^{(2)})^S[-h] \xleftarrow{(\mu'_2)^*} (\tilde{V} \otimes_S \mathcal{N}^{(2)})^S[-h+1] \xleftarrow{(\mu'_1)^*} (\mathcal{N}^{(2)})^S[-h+3] \leftarrow 0.
\end{aligned}$$

**Proposition 3.1** *We have  $\mu'_i = \pm(\mu'_{12-i})^*$ ,  $i = 1, \dots, 11$ .*

*Proof:* (i)  $\mu'_1 = -(\mu'_{11})^*$ : Let  $a \in V$ ,  $x \in A$  and  $y \in \mathcal{N}^{(2)}$ . Then

$$\begin{aligned}
(\mu'_1(a \otimes x), y) &= (xa - ax, y) = (x, ay - y\beta(a)) = (a \otimes x, - \sum_{b \in \mathcal{G}_1} \tilde{b} \otimes (y\beta(b) - by)) \\
&= (a \otimes x, -\mu'_{11}(y)).
\end{aligned}$$

(ii)  $\mu'_2 = (\mu'_{10})^*$ : Let  $a, a' \in V$ ,  $x \in A$  and  $y \in \mathcal{N}^{(2)}$ . Then

$$\begin{aligned} (\mu'_2(\tilde{a} \otimes x), \tilde{a}' \otimes y) &= \left( \sum_{b, b' \in \mathcal{G}_1} W_{abb'}(b' \otimes xb + b \otimes b'x), \tilde{a}' \otimes y \right) \\ &= \left( \sum_{b \in \mathcal{G}_1} W_{aba'}xb + \sum_{b' \in \mathcal{G}_1} W_{aa'b'}b'x, y \right) = \left( x, \sum_{b \in \mathcal{G}_1} W_{aba'}by + \sum_{b' \in \mathcal{G}_1} W_{aa'b'}y\beta(b') \right) \\ &= (\tilde{a} \otimes x, \sum_{b, b' \in \mathcal{G}_1} W_{b'ba'}(b' \otimes by + b \otimes y\beta(b'))) = (\tilde{a} \otimes x, \mu'_{10}(\tilde{a}' \otimes y)). \end{aligned}$$

(iii)  $\mu'_3 = -(\mu'_9)^*$ : Let  $a' \in V$ ,  $x \in A$  and  $y \in \mathcal{N}^{(2)}$ . Then

$$\begin{aligned} (\mu'_3(x), a' \otimes y) &= \left( \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes (xa - ax), a' \otimes y \right) = (xa' - a'x, y) = (x, a'y - y\beta(a')) \\ &= (x, -\mu'_9(a' \otimes y)). \end{aligned}$$

(iv)  $\mu'_4 = (\mu'_8)^*$ : Let  $x \in \mathcal{N}$  and  $y \in \mathcal{N}^{(2)}$ . Then

$$\begin{aligned} (\mu'_4(x), y) &= \left( \sum_j w_j^* \beta(x) w_j, y \right) = \left( \sum_j w_j x \beta^2(w_j^*), y \right) = \left( x, \sum_j \beta^2(w_j^*) y \beta(w_j) \right) \\ &= \left( x, \sum_j w_j^* \beta(y) \beta^2(w_j) \right) = (x, \mu'_8(y)), \end{aligned}$$

where the second equality holds since if  $\{w_j^*\}$  is a dual basis of  $\{w_j\}$ , then  $\{w_j\}$  is a dual basis of  $\{\beta^2(w_j^*)\}$ , and  $\sum_j w_j^* \beta(x) w_j = 0 = \sum_j w_j x \beta^2(w_j^*)$  unless  $|x| = 0$  such that  $\beta(x) = x$ . The penultimate equality is given by replacing the basis  $\{\beta^2(w_j^*)\}$  with the equivalent basis  $\{w_j^*\}$ , and the fact that  $\sum_j \beta^2(w_j^*) y \beta(w_j) = 0 = \sum_j w_j^* \beta(y) \beta^2(w_j)$  unless  $|y| = 0$  such that  $\beta(y) = y$ .

(v)  $\mu'_5 = -(\mu'_7)^*$ : Let  $a \in V$  and  $x, y \in \mathcal{N}$ . Then

$$\begin{aligned} (\mu'_5(a \otimes x), y) &= (x \beta^2(a) - ax, y) = (x, \beta^2(a)y - y\beta(a)) \\ &= (a \otimes x, -\sum_{b \in \mathcal{G}_1} \tilde{b} \otimes (y\beta^2(b) - by)) = (a \otimes x, -\mu'_7(y)). \end{aligned}$$

(vi)  $\mu'_6 = (\mu'_6)^*$ : Let  $a, a' \in V$  and  $x, y \in \mathcal{N}$ . Then

$$\begin{aligned} (\mu'_6(\tilde{a} \otimes x), \tilde{a}' \otimes y) &= \left( \sum_{b, b' \in \mathcal{G}_1} W_{abb'}(b' \otimes x\beta^2(b) + b \otimes b'x), \tilde{a}' \otimes y \right) \\ &= \left( \sum_{b \in \mathcal{G}_1} W_{aba'}x\beta^2(b) + \sum_{b' \in \mathcal{G}_1} W_{aa'b'}b'x, y \right) = \left( x, \sum_{b \in \mathcal{G}_1} W_{aba'}\beta^2(b)y + \sum_{b' \in \mathcal{G}_1} W_{aa'b'}y\beta(b') \right) \\ &= (\tilde{a} \otimes x, \sum_{b, b' \in \mathcal{G}_1} W_{b'ba'}(b' \otimes by + b \otimes y\beta^2(b'))) = (\tilde{a} \otimes x, \mu'_6(\tilde{a}' \otimes y)). \end{aligned}$$

□

Note however that  $(\mu'_{12})^* = \mu'_{12} \circ \beta$ : Let  $x, y \in A$ . Then

$$\begin{aligned} (\mu'_{12}(x), y) &= \left( \sum_j w_j^* \beta(x) \beta(w_j), y \right) = (\beta(x), \sum_j \beta(w_j) y \beta(w_j^*)) = (x, \sum_j w_j \beta^2(y) w_j^*) \\ &= \left( x, \sum_j w_j^* \beta^2(y) \beta(w_j) \right) = (x, \mu'_{12}(\beta(y))), \end{aligned}$$

where the penultimate equality holds since if  $\{w_j^*\}$  is a dual basis of  $\{w_j\}$ , then  $\{\beta(w_j)\}$  is a dual basis of  $\{w_j^*\}$ .

From the self-duality of the Hochschild homology complex (11) and  $(\mu'_{12})^* = \mu'_{12} \circ \beta$ , we have

$$\begin{aligned} HH_i(A)^* &\cong HH_{11-i}(A)[3h], & i = 1, \dots, 10, \\ HH_{11}(A)^* &\cong HH_{12}(A)[6h]. \end{aligned}$$

The reduced Hochschild homology  $\overline{HH}_\bullet(A)$  is defined as  $\overline{HH}_0(A) = HH_0(A)/S$  and  $\overline{HH}_n(A) = HH_n(A)$ ,  $n > 0$ .

### 3.2 The cyclic homology of $A(\mathcal{G}, W)$

Before we determine the Hochschild homology of  $A(\mathcal{G}, W)$  for certain  $SU(3)$   $\mathcal{ADE}$  graphs, we introduce cyclic homology. We begin by introducing the differential graded algebra  $\Omega^\bullet A$  of non-commutative forms of  $A$ , and the non-commutative de Rham homology.

The  $A$ - $A$  bimodule  $\Omega^1 A$  of non-commutative relative 1-forms on  $A$  is defined as the kernel of the multiplication map  $A \otimes_S A \rightarrow A$ . The differential graded algebra  $\Omega^\bullet A$  of non-commutative forms of  $A$  is obtained by taking tensor powers of  $\Omega^1 A$ . The graded commutator in  $\Omega^\bullet A$  is given by  $[\omega, \omega'] = \omega\omega' - (-1)^{|\omega||\omega'|}\omega'\omega$ , where  $|\omega| = n$  denotes the homological degree of  $\omega \in \Omega^n A$ . The reduced non-commutative de Rham homology of  $A$  is defined by

$$\overline{H}DR_n(A) := H_n(\Omega^\bullet A / (S + [\Omega^\bullet A, \Omega^\bullet A]), d),$$

where the natural differential  $\Omega^\bullet A \rightarrow \Omega^{\bullet+1} A$  descends to a de Rham differential on  $\Omega^\bullet A / (S + [\Omega^\bullet A, \Omega^\bullet A])$ .

Since  $A$  is an augmented  $S$ -algebra, i.e.  $A_0 = S$  and there is an augmentation  $\varphi : A \rightarrow S$  such that  $\varphi(1) = 1$ , by the non-commutative Poincaré lemma [27] (see also [30, Lemma 4.5]),  $\overline{H}DR_n(A) = \overline{H}DR_n(S) = 0$  for all  $n$ . Thus, from [16, Lemma 3.6.1], there is an exact sequence

$$0 \longrightarrow \overline{HH}_0(A) \xrightarrow{B} \overline{HH}_1(A) \xrightarrow{B} \overline{HH}_2(A) \xrightarrow{B} \overline{HH}_0(A) \longrightarrow \dots \quad (12)$$

where  $B$  is the Connes differential, which is degree-preserving, and the reduced cyclic homology of  $A$  can be defined by

$$\overline{HC}_n(A) = \ker(B : \overline{HH}_{n+1}(A) \rightarrow \overline{HH}_{n+2}(A)) = \text{Im}(B : \overline{HH}_n(A) \rightarrow \overline{HH}_{n+1}(A)).$$

The usual cyclic homology is related to the reduced cyclic homology by  $\overline{HC}_0(A) = HC_0(A)/S$  and  $\overline{HC}_n(A) = HC_n(A)$ ,  $n > 0$ .

The (graded) Euler characteristic of the reduced cyclic homology is the polynomial in  $t$  defined by  $\chi_{\overline{HC}(A)}(t) = \sum_{i=0}^{\infty} (-1)^i H_{\overline{HC}_i(A)}(t)$ . It turns out to be easier to describe the Euler characteristic of  $\text{Sym}(\overline{HC}(A))_+$ , where if  $\chi_{\overline{HC}(A)}(t) = \sum_{k=0}^{\infty} a_k t^k$  then  $\chi_{\text{Sym}(\overline{HC}(A))_+}(t) = \prod_{k=1}^{\infty} (1 - t^k)^{a_k}$ . In [16, Prop. 3.7.1] it was shown that for  $A$  the preprojective algebra of a non-Dynkin quiver,

$$\prod_{k=1}^{\infty} (1 - t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s), \quad (13)$$

where  $H_A(t)$  is the Hilbert series of  $A$ . The result (13) was extended to the case where  $A$  is a Calabi-Yau algebra of dimension 3 in [26, Prop. 5.4.9]. In the case when  $A$  is the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$ , the differential graded algebra  $\mathfrak{D}_\bullet = T_S(V \oplus V^* \oplus S^*)$  in [26, Prop. 5.4.9] is no longer exact. However, we can build a larger free differential graded algebra  $\mathfrak{D}'_\bullet$  by adding generators  $x_n \in \mathfrak{D}'_n$  whose images under the differential give a basis for  $H_n(\mathfrak{D}'_\bullet)$ , for each  $n > 0$ . These generators lie in degree  $nh$ , where  $h$  is the Coxeter number of  $\mathcal{G}$ . Then  $\mathfrak{D}'_\bullet$  gives a free resolution of  $A$ , and a correction term corresponding to the numerator  $1 - Pt^{hs}$  of  $H_A(t^s)$  appears in the formula (13). Thus the result (13) holds for the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$  (c.f. [15, Lemma 4.4.1] in the case where  $A$  is the preprojective algebra of a Dynkin quiver).

### 3.3 $HH_0(A)$ for $A = A(\mathcal{G}, W)$

In this section we compute the zeroth Hochschild homology  $HH_0(A) = \ker(\mu'_0)/\text{Im}(\mu'_1) = A/[A, A]$  for the simplest graphs, namely the graphs  $\mathcal{A}^{(n)}$ ,  $n \geq 4$ ,  $\mathcal{D}^{(3k+3)}$ ,  $k \geq 1$ ,  $\mathcal{A}^{(n)*}$ ,  $n \geq 5$ ,  $\mathcal{D}^{(n)*}$ ,  $n \geq 5$ ,  $\mathcal{E}^{(8)}$  and  $\mathcal{E}^{(8)*}$ .

For a graded algebra  $B = \bigoplus_{k=0}^{\infty} B_k$ , let  $B_+$  denote the positive degree part  $B_+ = \bigoplus_{k=1}^{\infty} B_k$ . For any  $a, b \in A_+$  such that  $r(a) = s(b)$  and  $s(a) \neq r(b)$ ,  $[a, b] = ab$ , thus any non-cyclic path  $ab$  is in  $[A, A]$ . For  $a, b \in A_+$  such that  $r(a) = s(b)$  and  $s(a) = r(b)$ ,  $[a, b] = ab - ba$ , thus cyclic paths are equivalent in  $A/[A, A]$  if one is a cyclic permutation of the other. Thus to determine  $A/[A, A]$  we first consider all cyclic paths in  $iAi$  for some  $i \in \mathcal{G}_0$ , then consider all cyclic paths in  $jAj$  which do not pass through the vertex  $i \neq j$ , for some  $j \in \mathcal{G}_0$ , and so on. Note that  $S \hookrightarrow A/[A, A]$  since  $[i, j] = 0$  for all  $i, j \in S$  and  $[a, b] \subset A_+$  if either  $a$  or  $b$  have non-zero length.

#### 3.3.1 The identity $\mathcal{A}^{(n)}$ graphs

The unique cell system  $W$  (up to equivalence) was computed in [21, Theorem 5.1]. For the graph  $\mathcal{A}^{(n)}$ ,  $n \geq 4$ , the space of cyclic paths  $(0, 0)A_+(0, 0) = 0$ . Thus for any vertex  $i \neq (0, 0)$ , any cyclic path  $x \in iA_+i$  which passes through  $(0, 0)$  is a cyclic permutation of a cyclic path  $x' \in (0, 0)A_+(0, 0)$ . Similarly, any cyclic path  $x \in iA_+i$  which does not pass through  $(0, 0)$  can be transformed by a combination the relations in  $A$  and cyclic permutations to a cyclic path  $x' \in (0, 0)A_+(0, 0)$ . Thus any cyclic path  $x \in iA_+i$  will be zero in  $A/[A, A]$ , and we obtain

$$HH_0(A) \cong S. \quad (14)$$

#### 3.3.2 The orbifold $\mathcal{D}^{(3k+3)}$ graphs

We now consider the graphs  $\mathcal{D}^{(3k+3)}$ ,  $k \geq 1$ , which are  $\mathbb{Z}_3$ -orbifolds of  $\mathcal{A}^{(3k+3)}$ . The graph  $\mathcal{D}^{(9)}$  is illustrated in Figure 3. The weights  $W(\Delta)$  for  $\mathcal{A}^{(3k+3)}$  are invariant under the  $\mathbb{Z}_3$  symmetry of the graph given by rotation by  $2\pi/3$ . Thus there is an orbifold solution for the cell system  $W$  on  $\mathcal{D}^{(3k+3)}$  where the weights  $W(\Delta)$  are given by the corresponding weights for  $\mathcal{A}^{(3k+3)}$  [21, Theorem 6.2]. More precisely, excluding triangles  $\Delta$  which contain one of the triplicated vertices  $(k, k)_l$ , the weight  $W(\Delta_{i_1, i_2, i_3})$  for the triangle  $\Delta_{i_1, i_2, i_3} = i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow i_1$  on  $\mathcal{D}^{(3k+3)}$  is given by the weight  $W(\Delta_{i_1^{(0)}, i_2^{(1)}, i_3^{(2)}}) = W(\Delta_{i_1^{(1)}, i_2^{(2)}, i_3^{(0)}}) = W(\Delta_{i_1^{(2)}, i_2^{(0)}, i_3^{(1)}})$  for  $\mathcal{A}^{(3k+3)}$ , where  $i_l^{(0)}, i_l^{(1)}, i_l^{(2)}$  are the three vertices of

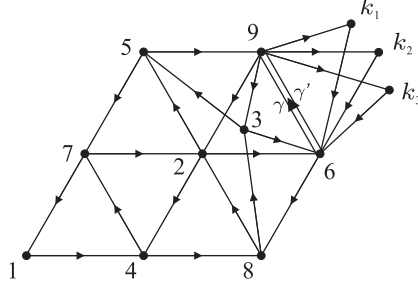


Figure 3: Graph  $\mathcal{D}^{(9)}$

$\mathcal{A}^{(3k+3)}$  which are identified under the  $\mathbb{Z}_3$  action to give the vertex  $i_l$  of  $\mathcal{D}^{(3k+3)}$ ,  $l = 1, 2, 3$ . If for a triangle  $\triangle_{i_1, i_2, i_3}$  on  $\mathcal{D}^{(3k+3)}$  there is no choice of vertices  $i_1^{(j_1)}, i_2^{(j_2)}, i_3^{(j_3)}$  on  $\mathcal{A}^{(3k+3)}$  which lie on a closed loop of length three  $i_1^{(j_1)} \rightarrow i_2^{(j_2)} \rightarrow i_3^{(j_3)} \rightarrow i_1^{(j_1)}$ , then we have  $W(\triangle_{i_1, i_2, i_3}) = 0$ . The weight  $W(\triangle)$  for a triangle  $\triangle$  which contain one of the triplicated vertices  $(k, k)_l$  is just given by one third of the weight for the corresponding triangle on  $\mathcal{A}^{(3k+3)}$ . Thus the relations for  $\mathcal{D}^{(3k+3)}$  are given precisely by the relations for  $\mathcal{A}^{(3k+3)}$ , except for the relations  $\rho_\gamma, \rho_{\gamma'}$ , which involve the triplicated vertices  $(k, k)_l$ .

Any cyclic path on  $\mathcal{A}^{(3k+3)}$  yields a cyclic paths on  $\mathcal{D}^{(3k+3)}$  by the above orbifold procedure. These cyclic paths will be zero in  $A/[A, A]$ , except for those which pass along the double edge of  $\mathcal{D}^{(3k+3)}$  – although these paths can be made to pass through  $(0, 0)$  in  $A'/[A', A']$  for  $A' = A(\mathcal{A}^{(3k+3)}, W)$ , when we do this for  $A = A(\mathcal{D}^{(3k+3)}, W)$  we obtain a cyclic path which passes through the vertex 1 of  $\mathcal{D}^{(3k+3)}$  which corresponds to  $(0, 0)$  on  $\mathcal{A}^{(3k+3)}$ , but also a linear combination of cyclic paths which do not pass through the vertex 1, due to the fact that relations involving the double edge are not of the form  $x = \lambda x'$  for basis paths  $x, x' \in A$ . There are also cyclic paths in  $A$  which do not come from cyclic paths in  $A'$  by the orbifold procedure. These paths must necessarily pass along the double edge  $(\gamma, \gamma')$  of  $\mathcal{D}^{(3k+3)}$ . Using the relations in  $A$  and cyclic permutations, we can transform any such cyclic path, necessarily of length  $3j$ ,  $j \in \mathbb{N}$ , due to the three-colourability of  $\mathcal{D}^{(3k+3)}$ , to a linear combination of cyclic basis paths  $[(i_1 i_2 k_l i_1)^j]$ ,  $l = 1, 2, 3$ , where  $i_1 = s(\gamma)$ ,  $i_2 = r(\gamma)$ ,  $k_l := (k, k)_l$ , and  $x^m$  denotes the path  $xx \cdots x$  ( $m$  times). These basis paths are not equivalent in  $[A, A]$ , except when  $j = k$  where  $[(i_1 i_2 k_1 i_1)^k] = [(i_1 i_2 k_2 i_1)^k] = [(i_1 i_2 k_3 i_1)^k]$ , thus

$$HH_0(A) \cong S \oplus C, \quad (15)$$

where the graded vector space  $C = \bigoplus_{j=1}^{k-1} \bigoplus_{l=1}^3 \mathbb{C}[(i_1 i_2 k_l i_1)^j] \oplus \mathbb{C}[(i_1 i_2 k_1 i_1)^k]$ , and has Hilbert series  $H_C(t) = \sum_{j=1}^{k-1} 3t^{3j} + t^{3k}$ .

### 3.3.3 The conjugate $\mathcal{A}^*$ graphs

The unique cell system  $W$  (up to equivalence) was computed in [21, Theorems 7.1, 7.3 & 7.4], and we use the same notation for the cells here. The  $\mathcal{A}^{(n)*}$  graphs are illustrated in [21, Figure 11]. We illustrate the cases  $n = 7, 8$  here in Figure 4. The numbering of the vertices of  $\mathcal{A}^{(2m+1)*}$  that we use here is the same as that in [23], but the reverse of that



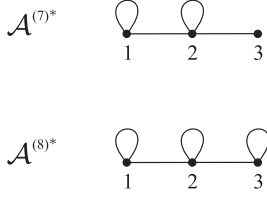


Figure 4: Graphs  $\mathcal{A}^{(7)*}$ ,  $\mathcal{A}^{(8)*}$

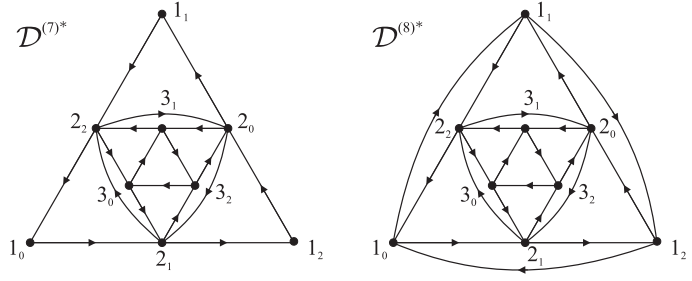


Figure 5: Graphs  $\mathcal{D}^{(7)*}$ ,  $\mathcal{D}^{(8)*}$

used in [21]. The relations in  $A(\mathcal{A}^{(n)*}, W)$  are

$$W_{112}[121] + W_{111}[111] = 0,$$

$$W_{p-1,p,p}[p(p-1)j] + W_{p,p,p}[ppp] + W_{p,p,p+1}[p(p+1)p] = 0, \quad (16)$$

$$W_{p,p,p+1}[pp(p+1)] + W_{p,p+1,p+1}[p(p+1)(p+1)] = 0, \quad (17)$$

$$W_{p,p,p+1}[(p+1)pp] + W_{p,p+1,p+1}[(p+1)(p+1)p] = 0, \quad (18)$$

where  $p = 2, \dots, r-1$  in (16), and  $p = 1, \dots, p'$  in (17), (18), where  $r = \lfloor (n-1)/2 \rfloor$ ,  $p' = r-1$  for even  $n$ , and  $p' = r-2$  for odd  $n$ . For even  $n$  we have the extra relation  $W_{r-1,r,r}[r(r-1)r] + W_{r,r,r}[rrr] = 0$ , and for odd  $n$  we have the extra relation  $[r(r-1)(r-1)] = [(r-1)(r-1)r] = 0$ .

We first consider the even case  $n = 2m + 2$ . Clearly all loops  $[pp]$  of length 1 are in  $A/[A, A]$ ,  $p = 1, \dots, m$ . Let  $d_l^p := \dim(pA_l p)$ . From the Hilbert series for  $A$ , we see that  $d_{2k}^p = d_{2k+1}^p = p = d_{2m-2k-1}^p = d_{2m-2k-2}^p$  if  $p \leq k$  or  $p \geq m - k + 1$ , and  $d_{2k}^p = d_{2k+1}^p = k + 1 = d_{2m-2k-1}^p = d_{2m-2k-2}^p$  if  $k + 1 \leq p \leq m - k$ , for  $2k \leq m - 1$ . Then  $\dim(A_{2k}^S) = \dim(A_{2k+1}^S) = (k + 1)(m - k) = \dim(A_{2m-2k-1}^S) = \dim(A_{2m-2k-2}^S)$  for  $2k \leq m - 1$ .

Each commutator of the form  $[[l(l+1)], [(l+1)l]] = [l(l+1)l] - [(l+1)l(l+1)]$  yields a relation between linearly independent paths of length 2 in  $A/[A, A]$ ,  $l = 1, \dots, m-1$ . There are  $m-1$  such relations, thus the dimension of  $(A/[A, A])_2$  is  $2(m-1) - (m-1) = m-1$ . Similarly the dimension of  $(A/[A, A])_3$  is  $m-1$ . Each commutator of the form  $C_p = [(p-1)ppp], [p(p-1)]$ ,  $p = 2, \dots, m$ , and  $C'_p = [(p-1)p(p+1)], [(p+1)p(p-1)]$ ,  $p = 2, \dots, m-1$ , yield relations between linearly independent paths of length 4 in  $A/[A, A]$ . There is one basis path in  $1A_4^S1$ , which we may take to be  $[11111]$ . Let  $w_1$  denote the basis element given by its image in  $A/[A, A]$ . Since  $d_4^2 = 2$ , the dimension of  $2A_4^S2$  is 2. However, the basis can be chosen such that one of the basis paths is identified with  $w_1$  in  $A/[A, A]$  by  $C_2$ , thus we obtain one new basis path  $w_2 \in (A/[A, A])_4$ , which may be chosen to be  $[22222]$ . Similarly, the dimension of  $pA_4^Sp$  is 3,  $p = 3, \dots, m-2$ , and the basis can be chosen such that two of the basis paths are identified with linear combinations of  $w_1, w_2, \dots, w_{p-1}$  in  $A/[A, A]$  by  $C_p$  and  $C'_p$ . Thus we obtain one new basis path  $w_p \in (A/[A, A])_4$  for each  $p = 3, \dots, m-2$ , which may be chosen to be  $[ppppp]$ . The dimension of  $(m-1)A_4^S(m-1)$  is 2, but by  $C_{m-1}, C'_{m-1}$  any such path can be identified with a linear combination of  $w_1, w_2, \dots, w_{m-2}$  in  $A/[A, A]$ . Similarly the single basis path in  $mA_4^Sm$  can be identified with a linear combination of  $w_1, w_2, \dots, w_{m-2}$  in  $A/[A, A]$ . Thus we obtain a basis  $\{w_1, \dots, w_{m-2}\}$  for  $(A/[A, A])_4$ . By a similar argument, we see that the dimension of  $(A/[A, A])_k$  is  $m - \lfloor k/2 \rfloor$  for all  $k = 0, 1, \dots, 2m-1$ , with basis

paths  $[ppp \cdots p]$ , for  $p = 1, \dots, m - \lfloor k/2 \rfloor$ . Thus

$$HH_0(A) \cong S \oplus C, \quad (19)$$

where the graded vector space  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{2m-1} (m - \lfloor j/2 \rfloor) t^j$ .

We now consider the odd case  $n = 2m + 1$ . Again, all loops  $[pp]$  of length 1 are in  $A/[A, A]$ , this time for  $p = 1, \dots, m - 1$  (note that there is no edge from vertex  $m$  to  $m$  on  $\mathcal{A}^{(2m+1)*}$ ). From the Hilbert series for  $A$ , we see that  $d_{2k}$  is given by the same formula as for the even case  $n = 2m + 2$ , for  $2k \leq m - 1$ , whilst  $d_{2k+1}^p = p = d_{2m-2k-2}^p$  if  $p \leq k$  or  $p \geq m - k$ ,  $d_{2k+1}^p = k + 1 = d_{2m-2k-3}^p$  if  $k + 1 \leq i \leq m - k - 1$ , and  $d_{2k+1}^m = 0$ , for  $2k \leq m - 2$ . Then  $\dim(A_{2k}^S) = (k + 1)(m - k) = \dim(A_{2m-2k-2}^S)$  for  $2k \leq m - 1$ , and  $\dim(A_{2k-1}^S) = (k + 1)(m - k - 1) = \dim(A_{2m-2k-3}^S)$  for  $2k \leq m - 2$ . By a similar argument as for the even case above, we see that the dimension of  $(A/[A, A])_k$  is  $m - \lfloor (k + 1)/2 \rfloor$  for all  $k = 0, 1, \dots, 2m - 2$ , with basis paths  $[ppp \cdots p]$ , for  $p = 1, \dots, m - \lfloor (k + 1)/2 \rfloor$ . Thus

$$HH_0(A) \cong S \oplus C, \quad (20)$$

where the graded vector space  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{2m-2} (m - \lfloor (j + 1)/2 \rfloor) t^j$ .

### 3.3.4 The conjugate orbifold $\mathcal{D}^*$ graphs

The graphs  $\mathcal{D}^{(n)*}$  are (three-colourable) unfolded versions, or  $\mathbb{Z}_3$ -orbifolds, of the graphs  $\mathcal{A}^{(n)*}$ , where we replace every vertex  $v$  of  $\mathcal{A}^{(n)*}$  by three vertices  $v_0, v_1, v_2$ , where  $v_a$  is of colour  $a$ , such that there are edges  $v_0 \rightarrow w_1, v_1 \rightarrow w_2$  and  $v_2 \rightarrow w_0$  if and only if there is an edge  $v \rightarrow w$  on  $\mathcal{A}^{(n)*}$ . The graphs  $\mathcal{D}^{(7)*}, \mathcal{D}^{(8)*}$  are illustrated in Figure 5.

Due to the three-colourability of the graph  $\mathcal{D}^{(n)*}$ , a closed loop on  $\mathcal{A}^{(n)*}$  will only be a closed loop on  $\mathcal{D}^{(n)*}$  if it has length  $3k, k \geq 0$ , and for each such closed loop on  $\mathcal{A}^{(n)*}$ , there are three corresponding closed loops of length  $3k$  on  $\mathcal{D}^{(n)*}$ . However, these three closed loops are identified in  $A/[A, A]$ , which can be seen as follows. As in the case of  $\mathcal{A}^{(n)*}$ ,  $(A/[A, A])_{3k}$  is generated by paths of the form  $[p_l p_{l+1} p_{l+2} p_l \cdots p_{l+3k}]$ , for  $l = 0, 1, 2 \bmod 3$  and  $p = 1, \dots, r$ , where  $r = m - \lfloor 3k/2 \rfloor$  for  $n = 2m + 2$  and  $r = m - \lfloor (3k + 1)/2 \rfloor$  for  $n = 2m + 1$ . Since  $[p_l p_{l+1} p_{l+2} p_l \cdots p_l]$  is a cyclic permutation of  $[p_{l+1} p_{l+2} p_l p_{l+1} \cdots p_{l+1}]$ , we see that for  $l = 0, 1, 2 \bmod 3$ , the cyclic paths  $[p_l p_{l+1} p_{l+2} p_l \cdots p_{l+3k}]$  are identified in  $A/[A, A]$ . Thus  $(A/[A, A])_{3k}$  has a basis given by  $[p_0 p_1 p_2 p_0 \cdots p_0]$ , for  $p = 1, \dots, r$ , and  $(A/[A, A])_{k'} = 0$  for  $k' \not\equiv 0 \bmod 3$ . Then

$$HH_0(A) \cong S \oplus C, \quad (21)$$

where for  $n = 2m + 2$  the graded vector space  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-1)/3 \rfloor} (m - \lfloor 3j/2 \rfloor) t^{3j}$ , whilst for  $n = 2m + 1$ ,  $H_C(t) = \sum_{j=1}^{\lfloor (2m-2)/3 \rfloor} (m - \lfloor (3j + 1)/2 \rfloor) t^{3j}$ .

### 3.3.5 The graph $\mathcal{E}^{(8)}$ for the conformal embedding $SU(3)_5 \subset SU(6)_1$

We now consider the graph  $\mathcal{E}^{(8)}$ , illustrated in Figure 6. The unique cell system  $W$  (up to equivalence) was computed in [21, Theorem 9.1]. The quotient algebra  $A$  has the relations, for  $l = 0, 1, 2$ ,

$$[1_l 2_{l+1} 3_{l+2}] = [2_{l+1} 3_{l+2} 1_l] = [3_{l+2} 1_l 2_{l+1}] = [4_{l+2} 3_l 2_{l+1}] = 0,$$

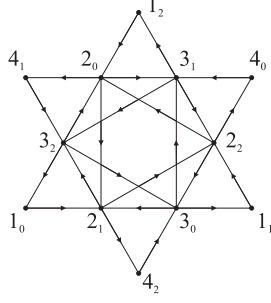


Figure 6: Graph  $\mathcal{E}^{(8)}$

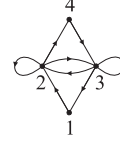


Figure 7: Graph  $\mathcal{E}^{(8)*}$

$$\begin{aligned}\sqrt{[3]}[2_l 2_{l+1} 2_{l+2}] &= -[2_l 3_{l+1} 2_{l+2}], & \sqrt{[3]}[3_l 3_{l+1} 3_{l+2}] &= [3_l 2_{l+1} 3_{l+2}], \\ \frac{-\sqrt{[3]}}{[2]}[3_l 1_{l+1} 2_{l+2}] &= [3_l 2_{l+1} 2_{l+2}] + [3_l 3_{l+1} 2_{l+2}], \\ \frac{-\sqrt{[3]}}{[2]}[2_l 4_{l+1} 3_{l+2}] &= [2_l 2_{l+1} 3_{l+2}] + [2_l 3_{l+1} 3_{l+2}].\end{aligned}$$

The only cyclic paths in  $A_+$  are of the form  $[2_l 2_{l+1} 2_{l+2} 2_l]$ ,  $[3_l 3_{l+1} 3_{l+2} 3_l]$ ,  $l = 1, \dots, 3$ . Now  $[2_l 2_{l+1} 2_{l+2} 2_l] = [2_{l+1} 2_{l+2} 2_l 2_{l+1}]$  by cyclic permutation, and  $\sqrt{[3]}[2_l 2_{l+1} 2_{l+2} 2_l] = -[2_l 3_{l+1} 2_{l+2} 2_l] = [3_{l+1} 2_{l+2} 2_l 3_{l+1}] = -[3_{l+1} 2_{l+2} 3_l 3_{l+1}] - (\sqrt{[2]}/\sqrt{[4]})[3_{l+1} 2_{l+2} 4_l 3_{l+1}] = -\sqrt{[3]}[3_{l+1} 3_{l+2} 3_l 3_{l+1}]$  in  $A/[A, A]$ , where the second and last equalities follow by cyclic permutation and the others follow from the relations in  $A$ . Thus we see that all cyclic paths in  $A_+$  are identified in  $A/[A, A]$  so that

$$HH_0(A) \cong S \oplus \mathbb{C}[2_0 2_1 2_2 2_0]. \quad (22)$$

### 3.3.6 The graph $\mathcal{E}^{(8)*}$ for the orbifold of the conformal embedding $SU(3)_5 \subset SU(6)_1 \rtimes \mathbb{Z}_3$

Consider the graph  $\mathcal{E}^{(8)*}$ , illustrated in Figure 7. The unique cell system  $W$  (up to equivalence) was computed in [21, Theorem 10.1]. The quotient algebra  $A$  has the relations

$$\begin{aligned}[123] &= [231] = [324] = [432] = 0, & [222] &= \frac{-1}{\sqrt{[3]}}[232], & [333] &= \frac{1}{\sqrt{[3]}}[323], \\ \frac{-\sqrt{[3]}}{[2]}[312] &= [322] + [332], & \frac{-\sqrt{[3]}}{[2]}[243] &= [223] + [233].\end{aligned}$$

Clearly the single edges  $[22]$ ,  $[33]$  are not in  $[A, A]$ , since the relations in  $A$  only change paths of length  $> 1$ , and edges are invariant under cyclic permutation. We have the relation  $\sqrt{[3]}[222 \cdots 2] = -[232 \cdots 2] = -[322 \cdots 23]$  for paths of length  $r$  in  $A/[A, A]$ ,  $2 \leq r \leq 5$ , where the first equality follows from the relation in  $A$  and the second follows by cyclic permutation. Thus in  $A/[A, A]$ , for  $r = 2$ , we obtain  $\sqrt{[3]}[222] = -[323] = -\sqrt{[3]}[333]$  by the relations in  $A$ . For  $r = 3$ , we have  $\sqrt{[3]}[2222] = -[3223] = [3323] + ([2]/\sqrt{[3]})[3123] = \sqrt{[3]}[3333]$ , by the relations in  $A$ , since the subpath  $[123] = 0$  in  $A$ . For  $r = 4$ , we have  $[3][22222] = -\sqrt{[3]}[32223] = [32323] = \sqrt{[3]}[33323] = [3][33333]$ , by the relations in  $A$ , but also  $\sqrt{[3]}[22222] = -[32223] = [33223] + ([2]/\sqrt{[3]})[31223] = -[33233] - ([2]/\sqrt{[3]})[33243] + ([2]/\sqrt{[3]})[23122] = -\sqrt{[3]}[33333]$ , by the relations in  $A$ , since the subpaths  $[324] = 0 = [123]$  in  $A$ , and we have used the cyclic permutation relation

in the penultimate equality. Then in  $A/[A, A]$ , we see that  $[22222] = 0 = [33333]$ . For  $r = 5$  we have  $[3][22222] = -\sqrt{[3]}[322223] = [323223] = \sqrt{[3]}[333223] = -[3][33333]$  in  $A/[A, A]$ , by the relations in  $A$ . Thus

$$HH_0(A) \cong S \oplus C, \quad (23)$$

where the graded vector space  $C = \mathbb{C}\{[22], [33]\} \oplus \mathbb{C}[222] \oplus \mathbb{C}[2222] \oplus \mathbb{C}[22222]$ , and has Hilbert series  $H_C(t) = 2t + t^2 + t^3 + t^5$ .

### 3.4 Determining the Hochschild homology of $A(\mathcal{G}, W)$ for trivial Nakayama automorphism

In this section we determine the Hochschild and cyclic homology for the graphs  $\mathcal{D}^{(3k)}$ ,  $k \geq 2$ ,  $\mathcal{A}^{(n)*}$ ,  $n \geq 4$ ,  $\mathcal{D}^{(3k)*}$ ,  $k \geq 2$ , and  $\mathcal{E}^{(8)*}$ . Here the almost Calabi-Yau algebra  $A$  has trivial Nakayama automorphism.

In this case, the Hochschild homology of  $A$  has minimal period at most 4, thus we have  $HH_i(A)^*[h] \cong HH_{3-i}(A)$ ,  $i = 1, 2$ , and  $HH_i(A)^* \cong HH_{7-i}(A)$ ,  $i = 3, 4$ . From the exactness of (12) we see that  $\ker(B : \overline{HH}_0(A) \rightarrow \overline{HH}_1(A)) = 0$ , and since the Connes differential  $B$  preserves degrees, we have  $\overline{HH}_1(A) \cong C \oplus X$ , for some graded vector space  $X$  which lives in degrees 1 to  $h-2$ . Then  $\overline{HH}_2(A) \cong C^*[h] \oplus X^*[h]$ , where  $C \cong \overline{HH}_0(A)$  and  $X^*[h]$  lives in degrees 2 to  $h-1$ . Now  $B : \overline{HH}_1(A) \rightarrow \overline{HH}_2(A)$  restricts to an isomorphism  $X \xrightarrow{\cong} X^*[h]$  since (12) is exact, and since it preserves degrees,  $X$  only lives in degrees 2 to  $h-2$ . A similar argument shows that  $\ker(B : \overline{HH}_0(A) \rightarrow \overline{HH}_1(A)) \cong X^*[h]$ , so that  $\overline{HH}_3(A) \cong C^*[h] \oplus K'$ , where the graded vector space  $K'$  lives in degrees 3 to  $h$ , and  $\overline{HH}_4(A) \cong C[h] \oplus K'^*[h]$ , where  $K'^*[h]$  lives in degrees  $h$  to  $2h-3$ . Since  $B : \overline{HH}_3(A) \rightarrow \overline{HH}_4(A)$  restricts to an isomorphism  $K' \xrightarrow{\cong} K'^*[h]$ , we see that  $K'$  lives only in degree  $h$ . We will write  $K' = K[h]$  where  $K$  is a vector space which lives in degree 0, so that  $\overline{HH}_3(A) \cong C^*[h] \oplus K[h]$  and  $\overline{HH}_4(A) \cong C[h] \oplus K^*[h]$ .

Thus for any almost Calabi-Yau algebra  $A$  with trivial Nakayama automorphism

$$\begin{array}{ccccc}
& 0 & & & \\
& \downarrow & & & \\
(\text{min deg, max deg}) & \overline{HH}_0(A) & \cong & C & \overline{HC}_0(A) \cong C \\
(0, h-3) & \downarrow B & \cong \downarrow & & \\
& \overline{HH}_1(A) & \cong & C \oplus X & \overline{HC}_1(A) \cong X \\
(1, h-2) & \downarrow B & \cong \downarrow & & \\
& \overline{HH}_2(A) & \cong & C^*[h] \oplus X^*[h] & \overline{HC}_2(A) \cong C^*[h] \\
(2, h-1) & \downarrow B & \cong \downarrow & & \\
& \overline{HH}_3(A) & \cong & C^*[h] \oplus K[h] & \overline{HC}_3(A) \cong K[h] \\
(3, h) & \downarrow B & \cong \downarrow & & \\
& \overline{HH}_4(A) & \cong & C[h] \oplus K^*[h] & \overline{HC}_4(A) \cong C[h] \\
(h, 2h-3) & \downarrow & & & \\
& \vdots & & & 
\end{array}$$

where  $X$  lives in degrees 2 to  $h-2$ ,  $K$  lives in degree 0, and  $\overline{HH}_{5+i}(A) \cong \overline{HH}_{1+i}(A)[h]$ ,  $\overline{HC}_{4+i}(A) \cong \overline{HC}_i(A)[h]$  for  $i \geq 0$ . Since  $C$  is known,  $X$  and  $K$  can be determined from the Euler characteristic  $\chi_{\overline{HC}(A)}(t)$  as they live in different degrees.

### 3.4.1 The graphs $\mathcal{D}^{(3k)}$

We consider the cases  $\mathcal{D}^{(6k)}$ ,  $\mathcal{D}^{(6k+3)}$  separately,  $k \geq 1$ . For the graph  $\mathcal{D}^{(6k)}$ ,  $k \geq 1$ ,  $\det(H_A(t)) = (1 - t^{6k})^{4(3k(k-1)+2)/2} (1 - t^{3k}) / (1 - t^3)^3$ , thus  $\chi_{\overline{HC}(A)}(t) = (\sum_{j=1}^{2k-1} 3t^{3j} - t^{3k} - (6k(k-1) + 2)t^{6k}) / (1 - t^{6k})$ . Then since  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{2k-2} 3t^{3j} + t^{6k-3}$ , we see that  $H_X(t) = t^3 + \sum_{j=2}^{2k-2} 3t^{3j} + t^{3k} + t^{6k-3}$  and  $H_K(t) = 6k(k-1) + 2$ , and we obtain:

**Theorem 3.2** *The Hochschild and cyclic homology of  $A = A(\mathcal{D}^{(6k)}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems constructed in [21], is given by*

$$\begin{aligned} HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\ HH_1(A) &\cong C \oplus X, & HC_1(A) &\cong X, \\ HH_2(A) &\cong C^*[6k] \oplus X^*[6k], & HC_2(A) &\cong C^*[6k], \\ HH_3(A) &\cong C^*[6k] \oplus K[6k], & HC_3(A) &\cong K[6k], \\ HH_4(A) &\cong C[6k] \oplus K^*[6k], & HC_4(A) &\cong C[6k], \\ HH_{4+i}(A) &\cong HH_i(A)[6k], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[6k], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $X$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{2k-2} 3t^{3j} + t^{6k-3}$ ,  $H_X(t) = t^3 + \sum_{j=2}^{2k-2} 3t^{3j} + t^{3k} + t^{6k-3}$  and  $H_K(t) = 6k(k-1) + 2$  respectively, where for  $k = 1$ ,  $H_X(t) = 0$ .

For  $\mathcal{D}^{(6k+3)}$ ,  $k \geq 1$ ,  $\det(H_A(t)) = (1 - t^{6k+3})^{6k^2+3} / (1 - t^3)^3$ , thus  $\chi_{\overline{HC}(A)}(t) = (\sum_{j=1}^{2k} 3t^{3j} - 6k^2 t^{6k+3}) / (1 - t^{6k+3})$ . Since  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{2k-1} 3t^{3j} + t^{6k}$ , we see that  $H_X(t) = t^3 + \sum_{j=2}^{2k-1} 3t^{3j} + t^{6k}$  and  $H_K(t) = 6k^2$ , and we obtain:

**Theorem 3.3** *The Hochschild and cyclic homology of  $A = A(\mathcal{D}^{(6k+3)}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems constructed in [21], is given by*

$$\begin{aligned} HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\ HH_1(A) &\cong C \oplus X, & HC_1(A) &\cong X, \\ HH_2(A) &\cong C^*[6k+3] \oplus X^*[6k+3], & HC_2(A) &\cong C^*[6k+3], \\ HH_3(A) &\cong C^*[6k+3] \oplus K[6k+3], & HC_3(A) &\cong K[6k+3], \\ HH_4(A) &\cong C[6k+3] \oplus K^*[6k+3], & HC_4(A) &\cong C[6k+3], \\ HH_{4+i}(A) &\cong HH_i(A)[6k+3], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[6k+3], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $X$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{2k-1} 3t^{3j} + t^{6k}$ ,  $H_X(t) = t^3 + \sum_{j=2}^{2k-1} 3t^{3j} + t^{6k}$  and  $H_K(t) = 6k^2$  respectively.

### 3.4.2 The $\mathcal{A}^*$ graphs

Let  $D_m$ ,  $D'_m$  denote the determinant of the denominator  $1 - \Delta_{\mathcal{G}}t + \Delta_{\mathcal{G}}^T t^2 - t^3$  of  $H_A(t)$  for  $\mathcal{G} = \mathcal{A}^{(2m+2)*}$ ,  $\mathcal{A}^{(2m+1)*}$  respectively, where  $\Delta_{\mathcal{G}}$  denotes the adjacency matrix of  $\mathcal{G}$ , and let  $T_1 = 1 - t + t^2 - t^3$ ,  $T_2 = t^2 - t$ . From the properties of determinants we can deduce the recursion relations  $D_m = T_1 D_{m-1} - T_2^2 D_{m-2}$  and  $D'_m = (1 - t^3) D_{m-1} - T_2^2 D_{m-2}$ , for  $m \geq 3$ , and  $D_1 = T_1$ ,  $D_2 = T_1^2 - T_2^2$ . It is easy to show by induction on  $m$  that  $D_m = (1-t)^m (1-t^{2m+2}) / (1-t^2)$ , and thus  $D'_m = (1-t)^{m-1} (1-t^{2m+1})$ . Then for  $\mathcal{A}^{(2m+2)*}$ ,

$\det H_A(t) = (1-t^{2m+2})^m D_m^{-1} = (1-t^2)(1-t^{2m+2})^{m-1}/(1-t)^m$ , thus  $\chi_{\overline{HC}(A)}(t) = (mt + (m-1)t^2 + mt^3 + (m-1)t^4 + \dots + (m-1)t^{2m} + mt^{2m+1})/(1-t^{2m+2})$ . Then  $H_X(t) = 0 = H_K(t)$ , i.e.  $X = 0 = K$ . For  $\mathcal{A}^{(2m+1)*}$ ,  $\det H_A(t) = (1-t^{2m+1})^m (D'_m)^{-1} = (1-t^{2m+1})^{m-1}/(1-t)^{m-1}$ , thus  $\chi_{\overline{HC}(A)}(t) = ((m-1)t + (m-1)t^2 + (m-1)t^3 + \dots + (m-1)t^{2m})/(1-t^{2m+1})$ . Then we again deduce that  $H_X(t) = 0 = H_K(t)$ , and we obtain:

**Theorem 3.4** *The Hochschild and cyclic homology of  $A = A(\mathcal{A}^{(n)*}, W)$ ,  $n \geq 4$ , where  $W$  is any cell system on  $\mathcal{A}^{(n)}$ , is given by*

$$\begin{aligned} HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\ HH_1(A) &\cong C, & HC_1(A) &= 0, \\ HH_2(A) &\cong C^*[n], & HC_2(A) &\cong C^*[n], \\ HH_3(A) &\cong C^*[n], & HC_3(A) &= 0, \\ HH_4(A) &\cong C[h], & HC_4(A) &\cong C[n], \\ HH_{4+i}(A) &\cong HH_i(A)[n], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[n], \quad i \geq 1, \end{aligned}$$

where the graded vector space  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{n-3} \lfloor (n-j-1)/2 \rfloor t^j$ .

### 3.4.3 The graph $\mathcal{D}^{(3k)*}$

The Nakayama automorphism is trivial for the graphs  $\mathcal{D}^{(3k)*}$ . We consider the cases  $\mathcal{D}^{(6k)*}$ ,  $\mathcal{D}^{(6k+3)*}$  separately. For the graph  $\mathcal{D}^{(6k)*}$ ,  $k \geq 1$ ,  $\det(H_A(t)) = (1-t^6)(1-t^{6k})^{9k-6}/(1-t^3)^{3k-1}$ , thus  $\chi_{\overline{HC}(A)}(t) = ((3k-1)t^3 + (3k-2)t^6 + (3k-1)t^9 + (3k-2)t^{12} + \dots + (3k-1)t^{6k-3} - (6k-4)t^{6k})/(1-t^{6k})$ . Then since  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-1)/3 \rfloor} (m - \lfloor 3j/2 \rfloor) t^{3j}$ , we have  $H_X(t) = 0$ ,  $H_K(t) = 6k - 4$ , and we obtain:

**Theorem 3.5** *The Hochschild and cyclic homology of  $A = A(\mathcal{D}^{(6k)*}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems constructed in [21], is given by*

$$\begin{aligned} HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\ HH_1(A) &\cong C, & HC_1(A) &= 0, \\ HH_2(A) &\cong C^*[6k], & HC_2(A) &\cong C^*[6k], \\ HH_3(A) &\cong C^*[6k] \oplus K[6k], & HC_3(A) &\cong K[6k], \\ HH_4(A) &\cong C[6k] \oplus K^*[6k], & HC_4(A) &\cong C[6k], \\ HH_{4+i}(A) &\cong HH_i(A)[6k], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[6k], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-1)/3 \rfloor} (m - \lfloor 3j/2 \rfloor) t^{3j}$  and  $H_K(t) = 6k - 4$  respectively.

For  $\mathcal{D}^{(6k+3)*}$ ,  $k \geq 1$ ,  $\det(H_A(t)) = (1-t^{6k+3})^{9k}/(1-t^3)^{3k}$ , thus  $\chi_{\overline{HC}(A)}(t) = (3kt^3 + 3kt^6 + \dots + 3kt^{6k} - 6kt^{6k+3})/(1-t^{6k+3})$ . Since  $C$  has Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-2)/3 \rfloor} (m - \lfloor (3j+1)/2 \rfloor) t^{3j}$ , we have  $H_X(t) = 0$ ,  $H_K(t) = 6k$ , and we obtain:

**Theorem 3.6** *The Hochschild and cyclic homology of  $A = A(\mathcal{D}^{(6k+3)*}, W)$ ,  $k \geq 1$ , where*



$W$  is equivalent to one of the cell systems constructed in [21], is given by

$$\begin{aligned}
HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\
HH_1(A) &\cong C, & HC_1(A) &\cong X, \\
HH_2(A) &\cong C^*[6k+3], & HC_2(A) &\cong C^*[6k+3], \\
HH_3(A) &\cong C^*[6k+3] \oplus K[6k+3], & HC_3(A) &\cong K[6k+3], \\
HH_4(A) &\cong C[6k+3] \oplus K^*[6k+3], & HC_4(A) &\cong C[6k+3], \\
HH_{4+i}(A) &\cong HH_i(A)[6k+3], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[6k+3], \quad i \geq 1,
\end{aligned}$$

where the graded vector spaces  $C, K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-2)/3 \rfloor} (m - \lfloor (3j+1)/2 \rfloor) t^{3j}$  and  $H_K(t) = 6k$  respectively.

### 3.4.4 The graph $\mathcal{E}^{(8)*}$

For the graph  $\mathcal{E}^{(8)*}$ ,  $\det(H_A(t)) = (1-t^2)(1-t^4)(1-t^8)^2/(1-t)^2$ , thus  $\chi_{\overline{HC}(A)}(t) = (2t+t^2+2t^3+2t^5+t^6+2t^7-2t^8)/(1-t^8)$ . Then  $H_X(t) = 0$ ,  $H_K(t) = 2$ , and we obtain:

**Theorem 3.7** *The Hochschild and cyclic homology of  $A = A(\mathcal{E}^{(8)*}, W)$ , where  $W$  is any cell system on  $\mathcal{E}^{(8)*}$ , is given by*

$$\begin{aligned}
HH_0(A) &\cong S \oplus C, & HC_0(A) &\cong S \oplus C, \\
HH_1(A) &\cong C, & HC_1(A) &= 0, \\
HH_2(A) &\cong C^*[8], & HC_2(A) &\cong C^*[8], \\
HH_3(A) &\cong C^*[8] \oplus K[8], & HC_3(A) &\cong K[8], \\
HH_4(A) &\cong C[8] \oplus K^*[8], & HC_4(A) &\cong C[8], \\
HH_{4+i}(A) &\cong HH_i(A)[8], \quad i \geq 1, & HC_{4+i}(A) &\cong HC_i(A)[8], \quad i \geq 1,
\end{aligned}$$

where the graded vector spaces  $C, K$  have Hilbert series  $H_C(t) = 2t + t^2 + t^3 + t^5$  and  $H_K(t) = 2$  respectively.

## 3.5 Determining the Hochschild homology of $A(\mathcal{G}, W)$ for non-trivial Nakayama automorphism

We now determine the Hochschild and cyclic homology for the graphs  $\mathcal{A}^{(n)}$ ,  $n = 4, 5, 6, 7$ ,  $\mathcal{E}^{(8)}$ . Here the almost Calabi-Yau algebra  $A$  has non-trivial Nakayama automorphism.

By a similar argument to that used in Section 3.4, for any almost Calabi-Yau algebra

$A$  with non-trivial Nakayama automorphism, we have

	$0$		
(min deg, max deg)	$\downarrow$		
$(0, h-3)$	$\overline{HH}_0(A) \cong C$		$\overline{HC}_0(A) \cong C$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(1, h-2)$	$\overline{HH}_1(A) \cong C$	$\oplus X_1$	$\overline{HC}_1(A) \cong X_1$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(2, h-1)$	$\overline{HH}_2(A) \cong X_2$	$\oplus X_1$	$\overline{HC}_2(A) \cong X_2$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(3, h)$	$\overline{HH}_3(A) \cong X_2$	$\oplus K_1[h]$	$\overline{HC}_3(A) \cong K_1[h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(h, 2h-3)$	$\overline{HH}_4(A) \cong X_3$	$\oplus K_1[h]$	$\overline{HC}_4(A) \cong X_3$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(h+1, 2h-2)$	$\overline{HH}_5(A) \cong X_3$	$\oplus X_4$	$\overline{HC}_5(A) \cong X_4$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(h+2, 2h-1)$	$\overline{HH}_6(A) \cong X_3^*[3h] \oplus X_4^*[3h]$		$\overline{HC}_6(A) \cong X_3^*[3h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(h+3, 2h)$	$\overline{HH}_7(A) \cong X_3^*[3h] \oplus K_1^*[2h]$		$\overline{HC}_7(A) \cong K_1^*[2h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(2h, 3h-3)$	$\overline{HH}_8(A) \cong X_2^*[3h] \oplus K_1^*[2h]$		$\overline{HC}_8(A) \cong X_2^*[3h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(2h+1, 3h-2)$	$\overline{HH}_9(A) \cong X_2^*[3h] \oplus X_1^*[3h]$		$\overline{HC}_9(A) \cong X_1^*[3h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(2h+2, 3h-1)$	$\overline{HH}_{10}(A) \cong C^*[3h] \oplus X_1^*[3h]$		$\overline{HC}_{10}(A) \cong C^*[3h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(2h+3, 3h)$	$\overline{HH}_{11}(A) \cong C^*[3h] \oplus K_2[3h]$		$\overline{HC}_{11}(A) \cong K_2[3h]$
	$\begin{smallmatrix} B \\ \downarrow \end{smallmatrix}$	$\cong \downarrow$	
$(3h, 4h-3)$	$\overline{HH}_{12}(A) \cong C[3h] \oplus K_2^*[3h]$		$\overline{HC}_{12}(A) \cong C[3h]$
	$\downarrow$		
	$\vdots$		

where  $X_1$  lives in degrees 2 to  $h-2$ ,  $X_2$  lives in degrees 3 to  $h-1$ ,  $X_3$  lives in degrees  $h+1$  to  $2h-3$ ,  $X_4$  lives in degrees  $h+2$  to  $2h-2$ ,  $K_i$  lives in degree 0,  $i = 1, 2$ , and  $\overline{HH}_{13+i}(A) \cong \overline{HH}_{1+i}(A)[3h]$ ,  $\overline{HC}_{12+i}(A) \cong \overline{HC}_i(A)[3h]$  for  $i \geq 0$ . The graded vector space  $K_1$  can be determined from the Euler characteristic  $\chi_{\overline{HC}(A)}(t)$  as it is the only vector space which lives in degree  $h$ . The vector spaces  $X_1, X_3$  can be determined by computing  $\overline{HH}_1(A), \overline{HH}_4(A)$  respectively. Then  $X_2, X_4, K_2$  can each be determined from knowledge of  $C \cong \overline{HH}_0(A)$ ,  $X_1, X_3$  and the Euler characteristic.

### 3.5.1 The $\mathcal{A}$ graphs

Here we determine the Hochschild and cyclic homology for the graphs  $\mathcal{A}^{(n)}$ ,  $n = 4, 5, 6, 7$ . The graphs  $\mathcal{A}^{(n)}$ ,  $n = 5, 6, 7$ , are illustrated in Figure 8. We have not yet been able to determine the Hochschild and cyclic homology for the case of general  $n$ .

We first consider the graph  $\mathcal{A}^{(4)}$ , for which  $\det(H_A(t)) = (1 - t^6)/(1 - t^3)$ . Thus  $\chi_{\overline{HC}(A)}(t) = (t^3 + t^6)/(1 - t^{12})$  and we see that  $H_{K_1}(t) = 0$ , and since  $C = 0$  for all the  $\mathcal{A}$

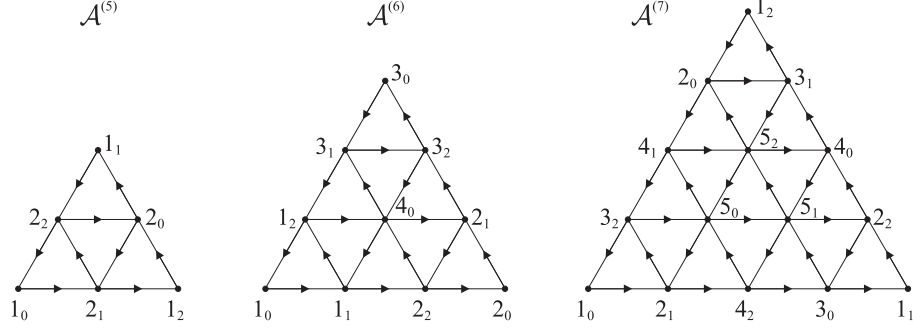


Figure 8: Graphs  $\mathcal{A}^{(n)}$ ,  $n = 5, 6, 7$

graphs,  $H_{K_2}(t) = 0$ . Since  $\ker(\mu'_1) \subset (V \otimes_S A)^S = 0$  and  $\ker(\mu'_4) = \mathcal{N}^S = 0$ , we see that  $\overline{HH}_1(A) = 0 = \overline{HH}_4(A)$ . Thus  $X_1 = X_3 = K_1 = 0$ , and from  $\chi_{\overline{HC}(A)}(t)$  we deduce that  $X_2$  has Hilbert series  $H_{X_2}(t) = t^3$  and  $X_4 = K_2 = 0$ .

**Theorem 3.8** *The Hochschild and cyclic homology of  $A = A(\mathcal{A}^{(4)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(4)}$ , is given by*

$$\begin{array}{ll}
HH_0(A) \cong S, & HC_0(A) \cong S, \\
HH_1(A) = 0, & HC_1(A) = 0, \\
HH_2(A) \cong X, & HC_2(A) \cong X, \\
HH_3(A) \cong X, & HC_3(A) = 0, \\
HH_4(A) = 0, & HC_4(A) = 0, \\
HH_5(A) = 0, & HC_5(A) = 0, \\
HH_6(A) = 0, & HC_6(A) = 0, \\
HH_7(A) = 0, & HC_7(A) = 0, \\
HH_8(A) \cong X^*[12], & HC_8(A) \cong X^*[12], \\
HH_9(A) \cong X^*[12], & HC_9(A) = 0, \\
HH_{10}(A) = 0, & HC_{10}(A) = 0, \\
HH_{11}(A) = 0, & HC_{11}(A) = 0, \\
HH_{12}(A) = 0, & HC_{12}(A) = 0, \\
HH_{12+i}(A) \cong HH_i(A)[12], \quad i \geq 1, & HC_{12+i}(A) \cong HC_i(A)[12], \quad i \geq 1,
\end{array}$$

where the graded vector space  $X$  has Hilbert series  $H_X(t) = t^3$ .

We now consider the graph  $\mathcal{A}^{(5)}$ , for which  $\det(H_A(t)) = (1 - t^{15})/(1 - t^3)$ . Thus  $\chi_{\overline{HC}(A)}(t) = (t^3 + t^6 + t^9 + t^{12})/(1 - t^{15})$  and we see that  $H_{K_1}(t) = 0$ , and since  $C = 0$  for all the  $\mathcal{A}$  graphs,  $H_{K_2}(t) = 0$ . We now explicitly determine  $\overline{HH}_1(A)$  and  $\overline{HH}_4(A)$ .

We begin with the graded vector space  $Y = \overline{HH}_1(A) = \ker(\mu'_1)/\text{Im}(\mu'_2)$ , and consider each graded piece  $Y_j$  separately. Due to the three-colourability of  $\mathcal{A}^{(5)}$ ,  $Y_j = 0$  for  $j = 1, 2$ . Thus we only need to determine  $Y_3$ . A basis for  $(\tilde{V} \otimes_S A)_2^S$  is given by the elements  $[2_{l+1}1_l] \otimes [1_l2_{l+1}]$ ,  $[1_{l+1}2_l] \otimes [2_l1_{l+1}]$  and  $[2_{l+1}2_l] \otimes [2_l2_{l+1}]$ , for  $l = 0, 1, 2 \bmod 3$ . We have  $\mu'_2([2_{l+1}1_l] \otimes [1_l2_{l+1}]) = W_{1_l2_{l+1}2_{l+2}}([2_{l+1}2_{l+2}] \otimes [2_{l+2}1_l2_{l+1}] + [1_l2_{l+1}] \otimes [2_{l+1}2_{l+2}1_l]) = W_{2_l2_{l+1}2_{l+2}}[2_{l+1}2_{l+2}] \otimes [2_{l+2}2_l2_{l+1}]$ , using the relations in  $A$ . Thus we see that  $[2_{l+1}2_{l+2}] \otimes [2_{l+2}2_l2_{l+1}] = 0$  in  $Y_3$ ,  $l = 0, 1, 2 \bmod 3$ . Thus  $Y_3 = 0$  and we obtain  $\overline{HH}_1(A) = 0$ . Since  $X_1 = 0$ , we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_2$  has Hilbert series  $H_{X_2}(t) = t^3$ .

We now consider  $Y' = \overline{HH}_4(A)$ , which lives in degrees 5 to 7. Now  $\ker(\mu'_4) = \mathcal{N}^S[5]$ , since  $\sum_j w_j^* \beta(x) w_j = 0$  for all  $x \in \mathcal{N}_+^S$  and  $\mathcal{N}_0^S = 0$ . As with  $\overline{HH}_1(A)$ ,  $Y'_j = 0$  for  $j = 5, 7$ , due to the three-colourability of  $\mathcal{A}^{(5)}$ . We now determine  $Y'_6$ . A basis for  $(V \otimes_S \mathcal{N})_1^S$  is given by  $[2_l 2_{l+1}] \otimes [2_{l+1}]$ ,  $l = 0, 1, 2 \bmod 3$ , and a basis for  $\mathcal{N}^S$  is given by  $[2_l 2_{l+1}]$ ,  $l = 0, 1, 2 \bmod 3$ . Now  $\mu'_5([2_l 2_{l+1}] \otimes [2_{l+1}]) = [2_{l+1} 2_{l+2}] - [2_l 2_{l+1}]$ , thus  $[2_0 2_1] = [2_1 2_2] = [2_2 2_0]$  in  $Y'_6$ . Then  $\overline{HH}_4(A) = \mathbb{C}[2_0 2_1][5] = X_3$ , and we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_4 = 0$ .

**Theorem 3.9** *The Hochschild and cyclic homology of  $A = A(\mathcal{A}^{(5)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(5)}$ , is given by*

$$\begin{array}{ll}
HH_0(A) \cong S, & HC_0(A) \cong S, \\
HH_1(A) = 0, & HC_1(A) = 0, \\
HH_2(A) \cong X_2, & HC_2(A) \cong X_2, \\
HH_3(A) \cong X_2, & HC_3(A) = 0, \\
HH_4(A) \cong X_3, & HC_4(A) \cong X_3, \\
HH_5(A) \cong X_3, & HC_5(A) = 0, \\
HH_6(A) \cong X_3^*[15], & HC_6(A) \cong X_3^*[15], \\
HH_7(A) \cong X_3^*[15], & HC_7(A) = 0, \\
HH_8(A) \cong X_2^*[15], & HC_8(A) \cong X_2^*[15], \\
HH_9(A) \cong X_2^*[15], & HC_9(A) = 0, \\
HH_{10}(A) = 0, & HC_{10}(A) = 0, \\
HH_{11}(A) = 0, & HC_{11}(A) = 0, \\
HH_{12}(A) = 0, & HC_{12}(A) = 0, \\
HH_{12+i}(A) \cong HH_i(A)[15], \quad i \geq 1, & HC_{12+i}(A) \cong HC_i(A)[15], \quad i \geq 1,
\end{array}$$

where the graded vector spaces  $X_2$  and  $X_3$  have Hilbert series  $H_{X_2}(t) = t^3$  and  $H_{X_3}(t) = t^6$ .

We now consider the graph  $\mathcal{A}^{(6)}$ , for which  $\det(H_A(t)) = (1-t^6)(1-t^9)(1-t^{18})/(1-t^3)$ . Thus  $\chi_{\overline{HC}(A)}(t) = (t^3 + t^{15} - 2t^{18})/(1-t^{18})$  and we see that  $H_{K_1}(t) = 0$ , and since  $C = 0$  for all the  $\mathcal{A}$  graphs,  $H_{K_2}(t) = 2$ . We now explicitly determine  $\overline{HH}_1(A)$  and  $\overline{HH}_4(A)$ .

We begin with the graded vector space  $Y = \overline{HH}_1(A)$ . Due to the three-colourability of  $\mathcal{A}^{(6)}$ ,  $Y_j = 0$  for  $j = 1, 2, 4$ , so we only need to determine  $Y_3$ . A basis for  $(\tilde{V} \otimes_S A)_2^S$  is given by the elements  $[i_{l+1} i_l] \otimes [i_l i_{l+1}]$ ,  $[(i+1)_2 i_1] \otimes [i_1 (i+1)_2]$ ,  $[i_1 4_0] \otimes [4_0 i_1]$ , and  $[4_0 i_2] \otimes [i_2 4_0]$ , for  $l = 0, 1, 2$ ,  $i = 1, 2, 3 \bmod 3$ . A basis for  $(V \otimes A)_3^S$  is given by  $[4_0 i_1] \otimes [i_1 i_2 4_0]$ ,  $[i_1 i_2] \otimes [i_2 4_0 i_1]$  and  $[i_2 4_0] \otimes [4_0 i_1 i_2]$ , for  $i = 1, 2, 3 \bmod 3$ . Under  $\mu'_2$  the basis elements of  $(\tilde{V} \otimes_S A)_2^S$  yield the following expressions after using the relations in  $A$ , for  $i = 1, 2, 3 \bmod 3$ :

$$\begin{aligned}
\mu'_2([i_1 i_0] \otimes [i_0 i_1]) &= -W_{4_0 i_1 i_2} [i_1 i_2] \otimes [i_2 4_0 i_1] = \mu'_2([i_0 i_2] \otimes [i_2 i_0]), \\
\mu'_2([i_2 i_1] \otimes [i_1 i_2]) &= W_{4_0 i_1 i_2} [4_0 i_1] \otimes [i_1 i_2 4_0] + W_{4_0 i_1 i_2} [i_2 4_0] \otimes [4_0 i_1 i_2], \\
\mu'_2([(i+1)_2 i_1] \otimes [i_1 (i+1)_2]) &= W_{4_0 i_1 (i+1)_2} [4_0 i_1] \otimes [i_1 (i+1)_2 4_0] \\
&\quad + W_{4_0 i_1 (i+1)_2} [(i+1)_2 4_0] \otimes [4_0 i_1 (i+1)_2], \\
\mu'_2([i_1 4_0] \otimes [4_0 i_1]) &= W_{4_0 i_1 i_2} [i_2 4_0] \otimes [4_0 i_1 i_2] + W_{4_0 i_1 (i+1)_2} [(i+1)_2 4_0] \otimes [4_0 i_1 (i+1)_2] \\
&\quad + W_{4_0 i_1 i_2} [i_1 i_2] \otimes [i_2 4_0 i_1], \\
\mu'_2([4_0 i_2] \otimes [i_2 4_0]) &= W_{4_0 i_1 i_2} [4_0 i_1] \otimes [i_1 i_2 4_0] + W_{4_0 (i-1)_1 i_2} [4_0 (i-1)_1] \otimes [(i-1)_1 i_2 4_0] \\
&\quad + W_{4_0 i_1 i_2} [i_1 i_2] \otimes [i_2 4_0 i_1].
\end{aligned}$$

Then from  $\text{Im}(\mu'_2)_3$  we obtain the following relations in  $Y_3$ :  $[i_1 i_2] \otimes [i_2 4_0 i_1] = 0$  and  $[4_0 i_1] \otimes [i_1 i_2 4_0] = -[i_2 4_0] \otimes [4_0 i_1 i_2] = (W_{4_0 3_1 3_2}/W_{4_0 i_1 i_2})[4_0 3_1] \otimes [3_1 1_2 4_0]$ .

We now consider  $\text{Ker}(\mu'_1)_3$ . Let  $x = \sum_{i=1}^3 (\lambda_i^0 [i_2 4_0] \otimes [4_0 i_1 i_2] + \lambda_i^1 [4_0 i_1] \otimes [i_1 i_2 4_0] + \lambda_i^2 [i_1 i_2] \otimes [i_2 4_0 i_1])$  be a general element in  $(V \otimes A)_3^S$ . Since  $\mu'_1(x) = \sum_{i=1}^3 ((\lambda_i^0 - \lambda_i^1)[4_0 i_1 i_2 4_0] + (\lambda_i^1 - \lambda_i^2)[i_1 i_2 4_0 i_1] + (\lambda_i^2 - \lambda_i^0)[i_2 4_0 i_1 i_2])$ , then  $x \in \text{Ker}(\mu'_1)$  if and only if  $\lambda_i^0 = \lambda_i^1 = \lambda_i^2$  for each  $i = 1, 2, 3$ . Using the relations from  $\text{Im}(\mu'_2)_3$ , a general element in  $Y_3$  is thus of the form  $\sum_i \lambda_i^0 ([i_2 4_0] \otimes [4_0 i_1 i_2] + [4_0 i_1] \otimes [i_1 i_2 4_0] + [i_1 i_2] \otimes [i_2 4_0 i_1]) = \sum_i \lambda_i^0 (-(W_{4_0 3_1 3_2}/W_{4_0 i_1 i_2})[4_0 3_1] \otimes [3_1 1_2 4_0] + 0 + (W_{4_0 3_1 3_2}/W_{4_0 i_1 i_2})[4_0 3_1] \otimes [3_1 1_2 4_0]) = 0$ . Thus  $Y_3 = 0$  and we obtain  $\overline{HH}_1(A) = 0$ . Since  $X_1 = 0$ , we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_2$  is a graded vector space with Hilbert series  $H_{X_2}(t) = t^3$ .

We now consider  $Y' = \overline{HH}_4(A)$ , which lives in degrees 6 to 9. Now  $\mathcal{N}_0^S = \mathbb{C}[4_0]$ , but  $\mu'_4(\lambda[4_0]) = 2\lambda[4_0 1_1 1_2 4_0]$ , thus  $\lambda[4_0] \in \text{ker}(\mu'_4)$  if and only if  $\lambda = 0$ . Thus  $\text{ker}(\mu'_4) = \mathcal{N}_+^S[6]$ , since  $\sum_j w_j^* \beta(x) w_j = 0$  for all  $x \in \mathcal{N}_+^S = \mathbb{C}[4_0 1_1 1_2 4_0]$ . As with  $\overline{HH}_1(A)$ ,  $Y'_j = 0$  for  $j = 7, 8$ , due to the three-colourability of  $\mathcal{A}^{(6)}$ , and  $Y'_6 = 0$  since  $\text{ker}(\mu'_4)_6 = 0$ . We now determine  $Y'_9$ . A basis for  $(V \otimes_S \mathcal{N})_1^S$  is given by  $[4_0 i_1] \otimes [i_1 i_2 4_0]$ ,  $[i_1(i+1)_2] \otimes [(i+1)_2 4_0(i-1)_1]$  and  $[i_2 4_0] \otimes [4_0(i-1)_1(i-1)_2]$ ,  $i = 0, 1, 2 \bmod 3$ . Now  $\mu'_5([4_0 1_1] \otimes [1_1 1_2 4_0]) = -[4_0 1_1 1_2 4_0]$ , thus  $\text{Im}(\mu'_5) = \mathcal{N}_+^S$ . Then  $Y'_9 = 0$  and we obtain  $\overline{HH}_4(A) = 0$ . Then since  $X_3 = 0$ , we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_4 = 0$ .

**Theorem 3.10** *The Hochschild and cyclic homology of  $A = A(\mathcal{A}^{(6)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(6)}$ , is given by*

$$\begin{array}{ll}
HH_0(A) \cong S, & HC_0(A) \cong S, \\
HH_1(A) = 0, & HC_1(A) = 0, \\
HH_2(A) \cong X, & HC_2(A) \cong X, \\
HH_3(A) \cong X, & HC_3(A) = 0, \\
HH_4(A) = 0, & HC_4(A) = 0, \\
HH_5(A) = 0, & HC_5(A) = 0, \\
HH_6(A) = 0, & HC_6(A) = 0, \\
HH_7(A) = 0, & HC_7(A) = 0, \\
HH_8(A) \cong X^*[18], & HC_8(A) \cong X^*[18], \\
HH_9(A) \cong X^*[18], & HC_9(A) = 0, \\
HH_{10}(A) = 0, & HC_{10}(A) = 0, \\
HH_{11}(A) \cong K[18], & HC_{11}(A) \cong K[18], \\
HH_{12}(A) \cong K^*[18], & HC_{12}(A) = 0, \\
HH_{12+i}(A) \cong HH_i(A)[18], \quad i \geq 1, & HC_{12+i}(A) \cong HC_i(A)[18], \quad i \geq 1,
\end{array}$$

where the graded vector spaces  $X$  and  $K$  have Hilbert series  $H_X(t) = t^3$  and  $H_K(t) = 2$ .

We now consider the graph  $\mathcal{A}^{(7)}$ , for which  $\det(H_A(t)) = (1 - t^{21})^3/(1 - t^3)$ . Thus  $\chi_{\overline{HC}(A)}(t) = (t^3 + t^6 + \dots + t^{18} - 2t^{21})/(1 - t^{21})$  and we see that  $H_{K_1}(t) = 0$ , and since  $C = 0$  for all the  $\mathcal{A}$  graphs,  $H_{K_2}(t) = 2$ . We now explicitly determine  $\overline{HH}_1(A)$  and  $\overline{HH}_4(A)$ .

We begin with the graded vector space  $Y = \overline{HH}_1(A)$ . Due to the three-colourability of  $\mathcal{A}^{(7)}$ ,  $Y_j = 0$  for  $j = 1, 2, 4, 5$ , so we only need to determine  $Y_3$ . A basis for  $(\tilde{V} \otimes_S A)_2^S$  is given by the elements  $[1_l 3_{l-1}] \otimes [3_{l-1} 1_l]$ ,  $[2_l 1_{l-1}] \otimes [1_{l-1} 2_l]$ ,  $[2_l 5_{l-1}] \otimes [5_{l-1} 2_l]$ ,  $[3_l 2_{l-1}] \otimes [2_{l-1} 3_l]$ ,  $[3_l 4_{l-1}] \otimes [4_{l-1} 3_l]$ ,  $[4_l 2_{l-1}] \otimes [2_{l-1} 4_l]$ ,  $[4_l 5_{l-1}] \otimes [5_{l-1} 4_l]$ ,  $[5_l 3_{l-1}] \otimes [3_{l-1} 5_l]$ ,  $[5_l 4_{l-1}] \otimes [4_{l-1} 5_l]$  and

$[5_l 5_{l-1}] \otimes [5_{l-1} 5_l]$ , for  $l = 0, 1, 2 \bmod 3$ . A basis for  $(V \otimes A)_3^S$  is given by  $[2_l 3_{l+1}] \otimes [3_{l+1} 5_{l+2} 2_l]$ ,  $[3_l 5_{l+1}] \otimes [5_{l+1} 2_{l+2} 3_l]$ ,  $[4_l 5_{l+1}] \otimes [5_{l+1} 2_{l+2} 4_l]$ ,  $[5_l 2_{l+1}] \otimes [2_{l+1} 3_{l+2} 5_l]$ ,  $[5_l 4_{l+1}] \otimes [4_{l+1} 3_{l+2} 5_l]$  and  $[5_l 5_{l+1}] \otimes [5_{l+1} 4_{l+2} 5_l]$ , for  $l = 0, 1, 2 \bmod 3$ . Under  $\mu'_2$  the basis elements of  $(\tilde{V} \otimes_S A)_2^S$  yield the following expressions after using the relations in  $A$ , for  $l = 0, 1, 2 \bmod 3$ :

$$\begin{aligned}
\mu'_2([1_l 3_{l-1}] \otimes [3_{l-1} 1_l]) &= W_{235}[2_{l+1} 3_{l-1}] \otimes [3_{l-1} 5_l 2_{l+1}] = \mu'_2([2_{l+1} 1_l] \otimes [1_l 2_{l+1}]), \\
\mu'_2([2_l 5_{l-1}] \otimes [5_{l-1} 2_l]) &= W_{235}[3_{l+1} 5_{l-1}] \otimes [5_{l-1} 2_l 3_{l+1}] + W_{245}[4_{l+1} 5_{l-1}] \otimes [5_{l-1} 2_l 4_{l+1}] \\
&\quad + W_{235}[2_{l+1} 3_{l-1}] \otimes [3_{l-1} 5_l 2_{l+1}], \\
\mu'_2([3_l 2_{l-1}] \otimes [2_{l-1} 3_l]) &= W_{235}[5_{l+1} 2_{l-1}] \otimes [2_{l-1} 3_l 5_{l+1}] + W_{235}[3_l 5_{l+1}] \otimes [5_{l+1} 2_{l-1} 3_l], \\
\mu'_2([3_l 4_{l-1}] \otimes [4_{l-1} 3_l]) &= W_{354}[5_{l+1} 4_{l-1}] \otimes [4_{l-1} 3_l 5_{l+1}] - W_{235}[3_l 5_{l+1}] \otimes [5_{l+1} 2_{l-1} 3_l], \\
\mu'_2([4_l 2_{l-1}] \otimes [2_{l-1} 4_l]) &= -W_{235}[5_{l+1} 2_{l-1}] \otimes [2_{l-1} 3_l 5_{l+1}] + W_{245}[4_l 5_{l+1}] \otimes [5_{l+1} 2_{l-1} 4_l], \\
\mu'_2([4_l 5_{l-1}] \otimes [5_{l-1} 4_l]) &= W_{455}[5_{l+1} 5_{l-1}] \otimes [5_{l-1} 4_l 5_{l+1}] - W_{245}[4_l 5_{l+1}] \otimes [5_{l+1} 2_{l-1} 4_l] \\
&\quad - W_{235}[3_{l+1} 5_{l-1}] \otimes [5_{l-1} 2_l 3_{l+1}], \\
\mu'_2([5_l 3_{l-1}] \otimes [3_{l-1} 5_l]) &= W_{354}[5_l 4_{l+1}] \otimes [4_{l+1} 3_{l-1} 5_l] + W_{235}[5_l 2_{l+1}] \otimes [2_{l+1} 3_{l-1} 5_l] \\
&\quad + W_{235}[2_{l+1} 3_{l-1}] \otimes [3_{l-1} 5_l 2_{l+1}], \\
\mu'_2([5_l 4_{l-1}] \otimes [4_{l-1} 5_l]) &= -W_{354}[5_{l+1} 4_{l-1}] \otimes [4_{l-1} 3_l 5_{l+1}] + W_{455}[5_l 5_{l+1}] \otimes [5_{l+1} 4_{l-1} 5_l] \\
&\quad - W_{235}[5_l 2_{l+1}] \otimes [2_l 3_{l-1} 5_l], \\
\mu'_2([5_l 5_{l-1}] \otimes [5_{l-1} 5_l]) &= -W_{245}[4_{l+1} 5_{l-1}] \otimes [5_{l-1} 2_l 4_{l+1}] - W_{354}[5_l 4_{l+1}] \otimes [4_{l+1} 3_{l-1} 5_l] \\
&\quad - W_{455}[5_{l+1} 5_{l-1}] \otimes [5_{l-1} 4_l 5_{l+1}] - W_{455}[5_l 5_{l+1}] \otimes [5_{l+1} 4_{l-1} 5_l],
\end{aligned}$$

where  $W_{ijk} := W_{i_0 j_1 k_2} = W_{i_1 j_2 k_0} = W_{i_2 j_0 k_1}$  for vertices  $i_l, j_l, k_l$  of  $\mathcal{A}^{(7)}$ ,  $l = 0, 1, 2$ . Then from  $\text{Im}(\mu'_2)_3$  we obtain the following relations in  $Y_3$ :

$$\begin{aligned}
[2_l 3_{l+1}] \otimes [3_{l+1} 5_{l+2} 2_l] &= 0, \quad l = 0, 1, 2 \bmod 3, & [5_0 5_1] \otimes [5_1 4_2 5_0] &= -x_2 - x_3, \\
[3_0 5_1] \otimes [5_1 2_2 3_0] &= -(W_{245}/W_{235})[4_0 5_1] \otimes [5_1 2_2 4_0] = -[5_1 2_2] \otimes [2_2 3_0 5_1] \\
&= (W_{354}/W_{235})[5_1 4_2] \otimes [4_2 3_0 5_1] = (W_{354}/W_{235})x_1 - (W_{455}/W_{235})x_2, \\
[3_l 5_2] \otimes [5_2 2_0 3_l] &= -(W_{245}/W_{235})[4_l 5_2] \otimes [5_2 2_0 4_l] = -[5_2 2_0] \otimes [2_0 3_l 5_2] = (W_{354}/W_{235})x_1, \\
[3_2 5_0] \otimes [5_0 2_1 3_2] &= -(W_{245}/W_{235})[4_2 5_0] \otimes [5_0 2_1 4_2] = -[5_0 2_1] \otimes [2_1 3_2 5_0] \\
&= (W_{354}/W_{235})[5_0 4_1] \otimes [4_1 3_2 5_0] = (W_{354}/W_{235})x_1 + (W_{455}/W_{235})x_3,
\end{aligned}$$

where  $x_1 = [5_2 4_0] \otimes [4_0 3_1 5_2]$ ,  $x_2 = [5_1 5_2] \otimes [5_2 4_0 5_1]$  and  $x_3 = [5_2 5_0] \otimes [5_0 4_1 5_2]$ .

We now consider  $\text{Ker}(\mu'_1)_3$ . Let  $x = \sum_{l=0}^2 (\lambda_1^l [3_l 5_{l+1}] \otimes [5_{l+1} 2_{l+2} 3_l] + \lambda_2^l [4_l 5_{l+1}] \otimes [5_{l+1} 2_{l+2} 4_l] + \lambda_3^l [5_l 2_{l+1}] \otimes [2_{l+1} 3_{l+2} 5_l] + \lambda_4^l [5_l 4_{l+1}] \otimes [4_{l+1} 3_{l+2} 5_l] + \lambda_5^l [5_l 5_{l+1}] \otimes [5_{l+1} 4_{l+2} 5_l])$  be a general element in  $(V \otimes A)_3^S$ . Now  $\mu'_1(x) = \sum_{l=0}^2 (\lambda_3^{l-1} [2_l 3_{l+1} 5_{l+2} 2_l] - \lambda_1^l [3_l 5_{l+1} 2_{l+2} 3_l] + (\lambda_1^{l-1} - \lambda_2^{l-1}(W_{235}/W_{245}) - \lambda_3^l + \lambda_4^l(W_{235}/W_{354}) + (\lambda_5^{l-1} - \lambda_5^l)(W_{235}/W_{455})) [5_l 2_{l+1} 3_{l+2} 5_l]) = 0$  if and only if  $\lambda_1^l = \lambda_3^l = 0$ ,  $\lambda_4^l = \lambda_2^{l-1} + (\lambda_5^l - \lambda_5^{l-1})(W_{354}/W_{455})$ , for  $l = 0, 1, 2 \bmod 3$ . Using the relations from  $\text{Im}(\mu'_2)_3$ , a general element in  $Y_3$  is thus of the form  $\sum_{l=0}^2 (\lambda_2^l [4_l 5_{l+1}] \otimes [5_{l+1} 2_{l+2} 4_l] + (\lambda_2^{l-1} + (\lambda_5^l - \lambda_5^{l-1})(W_{354}/W_{455})) [5_l 4_{l+1}] \otimes [4_{l+1} 3_{l+2} 5_l] + \lambda_5^l [5_l 5_{l+1}] \otimes [5_{l+1} 4_{l+2} 5_l]) = 0$ . Thus  $Y_3 = 0$  and we obtain  $\overline{HH}_1(A) = 0$ . Since  $X_1 = 0$ , we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_2$  is a graded vector space with Hilbert series  $H_{X_2}(t) = t^3 + t^6$ .

We now consider  $Y' = \overline{HH}_4(A)$ , which lives in degrees 7 to 11. Now  $\ker(\mu'_4) = \mathcal{N}^S[7]$ , since  $\sum_j w_j^* \beta(x) w_j = 0$  for all  $x \in \mathcal{N}_+^S$  and  $\mathcal{N}_0^S = 0$ . As with  $\overline{HH}_1(A)$ ,  $Y'_j = 0$  for  $j = 7, 8, 10, 11$ , due to the three-colourability of  $\mathcal{A}^{(7)}$ . We now determine  $Y'_9$ . A basis for  $(V \otimes_S \mathcal{N})_2^S$  is given by  $[4_l 5_{l+1}] \otimes [5_{l+1} 4_l]$ ,  $[5_l 4_{l+1}] \otimes [4_{l+1} 5_l]$  and  $[5_l 5_{l+1}] \otimes [5_{l+1} 5_l]$ ,



and a basis for  $\mathcal{N}^S$  is given by  $[4_l 5_{l+1} 4_{l+2}]$ ,  $[5_l 5_{l+1} 5_{l+2}]$ , for  $l = 0, 1, 2 \bmod 3$ . Using the relations in  $A$  we obtain  $\mu'_5([4_l 5_{l+1}] \otimes [5_{l+1} 4_l]) = -(W_{555}/W_{455})[5_{l+1} 5_{l+2} 5_l] - [4_l 5_{l+1} 4_{l+2}]$ ,  $\mu'_5([5_l 4_{l+1}] \otimes [4_{l+1} 5_l]) = (W_{555}/W_{455})[5_l 5_{l+1} 5_{l+2}] + [4_{l+1} 5_{l+2} 4_l]$  and  $\mu'_5([5_l 5_{l+1}] \otimes [5_{l+1} 5_l]) = [5_{l+1} 5_{l+2} 5_l] - [5_l 5_{l+1} 5_{l+2}]$ , for  $l = 0, 1, 2 \bmod 3$ . These yield the relations  $[4_l 5_{l+1} 4_{l+2}] = -(W_{555}/W_{455})[5_{l'} 5_{l'+1} 5_{l'+2}]$  in  $Y'_9$ , for all  $l, l' = 0, 1, 2 \bmod 3$ . Thus we obtain  $\overline{HH}_4(A) = \mathbb{C}[5_0 5_1 5_2][7]$ . Then  $X_3 = \mathbb{C}[5_0 5_1 5_2][7]$  and we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_4 = 0$ .

**Theorem 3.11** *The Hochschild and cyclic homology of  $A = A(\mathcal{A}^{(7)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(7)}$ , is given by*

$$\begin{array}{ll}
HH_0(A) \cong S, & HC_0(A) \cong S, \\
HH_1(A) = 0, & HC_1(A) = 0, \\
HH_2(A) \cong X_2, & HC_2(A) \cong X_2, \\
HH_3(A) \cong X_2, & HC_3(A) = 0, \\
HH_4(A) \cong X_3, & HC_4(A) \cong X_3, \\
HH_5(A) \cong X_3, & HC_5(A) = 0, \\
HH_6(A) \cong X_3^*[21], & HC_6(A) \cong X_3^*[21], \\
HH_7(A) \cong X_3^*[21], & HC_7(A) = 0, \\
HH_8(A) \cong X_2^*[21], & HC_8(A) \cong X_2^*[21], \\
HH_9(A) \cong X_2^*[21], & HC_9(A) = 0, \\
HH_{10}(A) = 0, & HC_{10}(A) = 0, \\
HH_{11}(A) \cong K[21], & HC_{11}(A) \cong K[21], \\
HH_{12}(A) \cong K^*[21], & HC_{12}(A) = 0, \\
HH_{12+i}(A) \cong HH_i(A)[21], \quad i \geq 1, & HC_{12+i}(A) \cong HC_i(A)[21], \quad i \geq 1,
\end{array}$$

where the graded vector spaces  $X_2$ ,  $X_3$  and  $K$  have Hilbert series  $H_{X_2}(t) = t^3 + t^6$ ,  $H_{X_3}(t) = t^9$  and  $H_K(t) = 2$ .

### 3.5.2 The graph $\mathcal{E}^{(8)}$

For the graph  $\mathcal{E}^{(8)}$ ,  $\det(H_A(t)) = (1 - t^6)(1 - t^{12})(1 - t^{24})^2 / (1 - t^3)^2 = \det(H_{A'}(t^3))$ , where  $A' = A(\mathcal{E}^{(8)*}, W)$ . Thus  $\chi_{\overline{HC}(A)}(t) = (2t^3 + t^6 + 2t^9 + 2t^{15} + t^{18} + 2t^{21} - 2t^{24}) / (1 - t^{24})$  and we see that  $H_{K_1}(t) = 0$ . Since  $C \cong \mathbb{C}[2_0 2_1 2_2 2_0]$  which lives in degree  $> 0$ , we see that  $H_{K_2}(t) = 2$ . We now explicitly determine  $\overline{HH}_1(A)$  and  $\overline{HH}_4(A)$ .

We begin with the graded vector space  $Y = \overline{HH}_1(A) = \ker(\mu'_1) / \text{Im}(\mu'_2)$ , and consider each graded piece  $Y_j$  separately. Due to the three-colourability of  $\mathcal{E}^{(8)}$ ,  $Y_j = 0$  for  $j = 1, 2, 4, 5$ . We will first determine  $Y_3$ . A basis for  $(\tilde{V} \otimes_S A)_3^S$  is given by  $[2_{l+1} 1_l] \otimes [1_l 2_{l+1}]$ ,  $[2_{l+1} 2_l] \otimes [2_l 2_{l+1}]$ ,  $[3_{l+1} 2_l] \otimes [2_l 3_{l+1}]$ ,  $[4_{l+1} 2_l] \otimes [2_l 4_{l+1}]$ ,  $[1_{l+1} 3_l] \otimes [3_l 1_{l+1}]$ ,  $[2_{l+1} 3_l] \otimes [3_l 2_{l+1}]$ ,  $[3_{l+1} 3_l] \otimes [3_l 3_{l+1}]$ , and  $[3_{l+1} 4_l] \otimes [4_l 3_{l+1}]$ , for  $l = 1, 2, 3$ . A basis for  $(V \otimes_S A)_3^S$  is given by  $[2_{l-1} 2_l] \otimes [2_l 2_{l+1} 2_{l-1}]$ ,  $[3_{l-1} 3_l] \otimes [3_l 3_{l+1} 3_{l-1}]$ ,  $[3_{l-1} 2_l] \otimes [2_l 2_{l+1} 3_{l-1}]$ ,  $[2_{l-1} 3_l] \otimes [3_l 3_{l+1} 2_{l-1}]$ ,  $[3_{l-1} 2_l] \otimes [2_l 3_{l+1} 3_{l-1}]$  and  $[2_{l-1} 3_l] \otimes [3_l 2_{l+1} 2_{l-1}]$ ,  $l = 1, 2, 3$ . Under  $\mu'_2$ ,  $[2_{l+1} 1_l] \otimes [1_l 2_{l+1}]$  gives

$$\begin{aligned}
\mu'_2([2_{l+1} 1_l] \otimes [1_l 2_{l+1}]) &= \sqrt{[2][3]}([2_{l+1} 3_{l-1}] \otimes [3_{l-1} 1_l 2_{l+1}] + [1_l 2_{l+1}] \otimes [2_{l+1} 3_{l-1} 1_l]) \\
&= -\sqrt{[3][4]}([2_{l+1} 3_{l-1}] \otimes [3_{l-1} 3_l 2_{l+1}] + [2_{l+1} 3_{l-1}] \otimes [3_{l-1} 2_l 2_{l+1}]),
\end{aligned}$$

using the relations in  $A$ . We get the same result from considering  $\mu'_2([1_l 3_{l-1}] \otimes [3_{l-1} 1_l])$ . We also obtain (up to some scalar factor)

$$\begin{aligned} \mu'_2([3_{l-1} 2_{l+1}] \otimes [2_{l+1} 3_{l-1}]) &= [3_l 2_{l+1}] \otimes [2_{l+1} 3_{l-1} 3_l] + [3_{l-1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1}] \\ &\quad + [2_l 2_{l+1}] \otimes [2_{l+1} 3_{l-1} 2_l] + [3_{l-1} 2_l] \otimes [2_l 2_{l+1} 3_{l-1}], \end{aligned}$$

and the results for  $\mu'_2([3_{l+1} 4_l] \otimes [4_l 3_{l+1}])$ ,  $\mu'_2([4_l 2_{l-1}] \otimes [2_{l-1} 4_l])$  and  $\mu'_2([2_{l-1} 3_{l+1}] \otimes [3_{l+1} 2_{l-1}])$  are given by the above results by interchanging  $1_p \leftrightarrow 4_p$ ,  $2_p \leftrightarrow 3_p$  for  $p = l, l+1, l-1$ . Finally, we also have (again up to some scalar factor)

$$\begin{aligned} \mu'_2([2_{l-1} 2_{l+1}] \otimes [2_{l+1} 2_{l-1}]) &= [3_l 2_{l+1}] \otimes [2_{l+1} 2_{l-1} 3_l] + [2_{l-1} 3_l] \otimes [3_l 2_{l+1} 2_{l-1}] \\ &\quad + \sqrt{[3]} [2_l 2_{l+1}] \otimes [2_{l+1} 2_{l-1} 2_l] + \sqrt{[3]} [2_{l-1} 3_l] \otimes [3_l 2_{l+1} 2_{l-1}], \\ \mu'_2([3_{l-1} 3_{l+1}] \otimes [3_{l+1} 3_{l-1}]) &= [2_l 3_{l+1}] \otimes [3_{l+1} 3_{l-1} 2_l] + [3_{l-1} 2_l] \otimes [2_l 3_{l+1} 3_{l-1}] \\ &\quad - \sqrt{[3]} [3_l 3_{l+1}] \otimes [3_{l+1} 3_{l-1} 3_l] - \sqrt{[3]} [3_{l-1} 2_l] \otimes [2_l 3_{l+1} 3_{l-1}]. \end{aligned}$$

Then from  $\text{Im}(\mu'_2)_3$  we obtain the following relations in  $Y_3$ :  $[2_0 2_1] \otimes [2_1 2_2 2_0] = [2_{l-1} 2_l] \otimes [2_l 2_{l+1} 2_{l-1}] = -[3_{l-1} 3_l] \otimes [3_l 3_{l+1} 3_{l-1}] = [3_{l-1} 2_l] \otimes [2_l 2_{l+1} 3_{l-1}] = -[2_{l-1} 3_l] \otimes [3_l 3_{l+1} 2_{l-1}] = -[3_{l-1} 2_l] \otimes [2_l 3_{l+1} 3_{l-1}] = [2_{l-1} 3_l] \otimes [3_l 2_{l+1} 2_{l-1}]$ ,  $l = 1, 2, 3$ , and thus  $Y_3 = \mathbb{C}[2_0 2_1] \otimes [2_1 2_2 2_0] \cong C$ .

We now determine  $Y_6$ . A basis for  $(\tilde{V} \otimes_S A)_6^S$  is given by  $[2_{l+1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 2_{l+1}]$ ,  $[3_{l+1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 3_{l+1}]$ ,  $[2_{l+1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 2_{l+1}]$  and  $[3_{l+1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 3_{l+1}]$ ,  $l = 1, 2, 3$ . For  $l = 1, 2, 3$ , we have (up to some scalar)

$$\begin{aligned} \mu'_2([2_{l+1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 2_{l+1}]) &= [2_{l-1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 3_{l+1} 2_{l-1}] \\ &\quad + [2_{l+1} 2_{l-1}] \otimes [2_{l-1} 3_l 2_{l+1} 2_{l-1} 3_l 2_{l+1}], \\ \mu'_2([3_{l+1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 3_{l+1}]) &= [3_{l-1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 2_{l+1} 3_{l-1}] \\ &\quad + [3_{l+1} 3_{l-1}] \otimes [3_{l-1} 2_l 3_{l+1} 3_{l-1} 2_l 3_{l+1}], \\ \mu'_2([2_{l+1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 2_{l+1}]) &= [3_{l-1} 3_l] \otimes [3_l 2_{l+1} 3_{l-1} 3_l 2_{l+1} 3_{l-1}] \\ &\quad - [2_{l+1} 2_{l-1}] \otimes [2_{l-1} 3_l 2_{l+1} 2_{l-1} 3_l 2_{l+1}], \\ \mu'_2([3_{l+1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 3_{l+1}]) &= [2_{l-1} 2_l] \otimes [2_l 3_{l+1} 2_{l-1} 2_l 3_{l+1} 2_{l-1}] \\ &\quad - [3_{l+1} 3_{l-1}] \otimes [3_{l-1} 2_l 3_{l+1} 3_{l-1} 2_l 3_{l+1}], \end{aligned}$$

which yield  $\text{Im}(\mu'_2)_6 = (\tilde{V} \otimes_S A)_6^S$ . Thus  $Y_6 = 0$ , and we obtain  $\overline{HH}_1(A) \cong C$ . Then  $X_1 = 0$  and from  $\chi_{\overline{HC}(A)}(t)$  we deduce that  $X_2$  is a graded vector space with Hilbert series  $H_{X_2}(t) = t^3 + t^6$ .

We now consider  $Y' = \overline{HH}_4(A)$ , which lives in degrees 8 to 13. Now  $\ker(\mu'_4) = \mathcal{N}^S[8]$ , since  $\sum_j w_j^* \beta(x) w_j = 0$  for all  $x \in \mathcal{N}_+^S$  and  $\mathcal{N}_0^S = 0$ . As with  $\overline{HH}_1(A)$ ,  $Y'_j = 0$  for  $j = 8, 10, 11, 13$ , due to the three-colourability of  $\mathcal{E}^{(8)}$ . We now determine  $Y'_9$ . A basis for  $(V \otimes_S \mathcal{N})_1^S$  is given by  $[2_l 2_{l+1}] \otimes [2_{l+1}]$  and  $[3_l 3_{l+1}] \otimes [3_{l+1}]$ ,  $l = 1, 2, 3$ , and a basis for  $\mathcal{N}_1^S$  is given by  $[2_l 2_{l+1}]$  and  $[3_l 3_{l+1}]$ ,  $l = 1, 2, 3$ . Now  $\mu'_5([2_l 2_{l+1}] \otimes [2_{l+1}]) = [2_{l+1} 2_{l-1}] - [2_l 2_{l+1}]$  and  $\mu'_5([3_l 3_{l+1}] \otimes [3_{l+1}]) = [3_{l+1} 3_{l-1}] - [3_l 3_{l+1}]$ ,  $l = 1, 2, 3$ , thus  $Y'_9 = (\mathbb{C}[2_0 2_1] \oplus \mathbb{C}[3_0 3_1])[8]$ . We now determine  $Y'_{12}$ . A basis for  $\mathcal{N}_4^S$  is given by  $[2_l 3_{l+1} 2_{l-1} 2_l 2_{l+1}]$  and  $[3_l 2_{l+1} 3_{l-1} 3_l 3_{l+1}]$ ,  $l = 1, 2, 3$ . Since  $\mu'_5([1_l 2_{l+1}] \otimes [2_{l+1} 3_{l-1} 3_l 1_{l+1}]) = [2_{l+1} 3_{l-1} 2_l 2_{l+1} 2_{l-1}]$  up to some scalar, by using the relations in  $A$ , and similarly  $\mu'_5([4_l 3_{l+1}] \otimes [3_{l+1} 2_{l-1} 2_l 4_{l+1}]) = [3_{l+1} 2_{l-1} 3_l 3_{l+1} 3_{l-1}]$ ,

$l = 1, 2, 3$  we see that  $Y'_{12} = 0$ . Thus  $\overline{HH}_4(A) = (\mathbb{C}[2_0 2_1] \oplus \mathbb{C}[3_0 3_1])[8]$ , and we obtain  $X_3 = (\mathbb{C}[2_0 2_1] \oplus \mathbb{C}[3_0 3_1])[8]$  and we deduce from  $\chi_{\overline{HC}(A)}(t)$  that  $X_4 = 0$ .

To summarize:

**Theorem 3.12** *The Hochschild and cyclic homology of  $A = A(\mathcal{E}^{(8)}, W)$ , where  $W$  is any cell system on  $\mathcal{E}^{(8)}$ , is given by*

$$\begin{array}{ll}
HH_0(A) \cong S \oplus C, & HC_0(A) \cong S \oplus C, \\
HH_1(A) \cong C, & HC_1(A) = 0, \\
HH_2(A) \cong X_2, & HC_2(A) \cong X_2, \\
HH_3(A) \cong X_2, & HC_3(A) = 0, \\
HH_4(A) \cong X_3, & HC_4(A) \cong X_3, \\
HH_5(A) \cong X_3, & HC_5(A) = 0, \\
HH_6(A) \cong X_3^*[24], & HC_6(A) \cong X_3^*[24], \\
HH_7(A) \cong X_3^*[24], & HC_7(A) = 0, \\
HH_8(A) \cong X_2^*[24], & HC_8(A) \cong X_2^*[24], \\
HH_9(A) \cong X_2^*[24], & HC_9(A) = 0, \\
HH_{10}(A) \cong C^*[24], & HC_{10}(A) \cong C^*[24], \\
HH_{11}(A) \cong C^*[24] \oplus K[24], & HC_{11}(A) \cong K[24], \\
HH_{12}(A) \cong C[24] \oplus K^*[24], & HC_{12}(A) \cong C[24], \\
HH_{12+i}(A) \cong HH_i(A)[24], \quad i \geq 1, & HC_{12+i}(A) \cong HC_i(A)[24], \quad i \geq 1,
\end{array}$$

where the graded vector spaces  $C$ ,  $X_2$ ,  $X_3$  and  $K$  have Hilbert series  $H_C(t) = t^3$ ,  $H_{X_2}(t) = t^3 + t^6$ ,  $H_{X_3}(t) = 2t^9$  and  $H_K(t) = 2$  respectively.

## 4 The Hochschild cohomology of $A(\mathcal{G}, W)$

### 4.1 The Hochschild cohomology complex

In this section we will construct a complex which determines the Hochschild cohomology of the almost Calabi-Yau algebra  $A = A(\mathcal{G}, W)$ . Each four-term piece of this complex will be identified up to a shift in degree with a four-term piece in the Hochschild homology complex (11).

The Hochschild cohomology  $HH^\bullet(A)$  of  $A$  may be defined as the derived functor  $HH^n(A) = \text{Ext}_{A^e}^n(A, A)$ , that is, the homology of the complex

$$0 \rightarrow \text{Hom}_{A^e}(P_0, A) \rightarrow \text{Hom}_{A^e}(P_1, A) \rightarrow \text{Hom}_{A^e}(P_2, A) \rightarrow \cdots$$

where  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$  is any projective resolution of  $A$ .

Following [15], we can make identifications  $\text{Hom}_{A^e}(A \otimes_S \mathcal{N}^{(k)}, A) = (\mathcal{N}^{(-k)})^S$ ,  $k = 0, 1, 2$ , by identifying  $\phi \in \text{Hom}_{A^e}(A \otimes_S \mathcal{N}^{(k)}, A)$  with the image  $\phi(1 \otimes 1) = x \in (\mathcal{N}^{(-k)})^S$ . We write  $\phi = x \circ - : A \otimes_S \mathcal{N}^{(-k)} \rightarrow A$ , and have  $\phi(y \otimes z) = x \circ (y \otimes z) = yx\beta^k(z)$ , for  $x \in (\mathcal{N}^{(-k)})^S$ ,  $y \in A$ ,  $z \in \mathcal{N}^{(k)}$ . We also make identifications  $\text{Hom}_{A^e}(A \otimes_S V \otimes_S \mathcal{N}^{(k)}, A) = (\tilde{V} \otimes_S \mathcal{N}^{(-k)})^S[-2]$ ,  $k = 0, 1, 2$ , by identifying  $\phi \in \text{Hom}_{A^e}(A \otimes_S V \otimes_S \mathcal{N}^{(k)}, A)$  which maps  $1 \otimes a \otimes 1 \mapsto x_a$  with the element  $\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a \in (\tilde{V} \otimes_S \mathcal{N}^{(-k)})^S$ . We write  $\phi = \sum_{b \in \mathcal{G}_1} \tilde{b} \otimes x_b \circ - : A \otimes_S V \otimes_S \mathcal{N}^{(-k)} \rightarrow A$ , and have  $\phi(y \otimes a \otimes z) = \sum_{b \in \mathcal{G}_1} \tilde{b} \otimes x_b \circ$

$(y \otimes a \otimes z) = yx_a\beta^k(z)$ , for  $\tilde{a} \otimes x_a \in (\tilde{V} \otimes_S \mathcal{N}^{(-k)})^S$ ,  $y \in A$ ,  $z \in \mathcal{N}^{(k)}$ . Similarly, we identify  $\text{Hom}_{A^e}(A \otimes_S \tilde{V} \otimes_S \mathcal{N}^{(k)}, A) = (V \otimes_S \mathcal{N}^{(-k)})^S[-2]$ ,  $k = 0, 1, 2$ , by identifying  $\phi$  which maps  $1 \otimes \tilde{a} \otimes 1 \mapsto y_a$  with the element  $\sum_{a \in \mathcal{G}_1} a \otimes y_a$ . We write  $\phi = \sum_{b \in \mathcal{G}_1} b \otimes y_b \circ - : A \otimes_S \tilde{V} \otimes_S \mathcal{N}^{(-k)} \rightarrow A$ , and have  $\phi(y \otimes \tilde{a} \otimes z) = yy_a\beta^k(z)$ , for  $a \otimes y_a \in (V \otimes_S \mathcal{N}^{(-k)})^S$ ,  $y \in A$ ,  $z \in \mathcal{N}^{(k)}$ .

Applying the functor  $\text{Hom}_{A^e}(-, A)$  to the periodic resolution (10) we get the Hochschild cohomology complex:

$$\begin{aligned} (\mathcal{N}^{(2)})^S[-h] &\xleftarrow{\mu_4^*} A^S[-3] \xleftarrow{\mu_3^*} (V \otimes_S A)^S[-3] \xleftarrow{\mu_2^*} (\tilde{V} \otimes_S A)^S[-2] \xleftarrow{\mu_1^*} A^S \leftarrow 0 \\ \mathcal{N}^S[-2h] &\xleftarrow{\mu_8^*} (\mathcal{N}^{(2)})^S[-h-3] \xleftarrow{\mu_7^*} (V \otimes_S \mathcal{N}^{(2)})^S[-h-3] \xleftarrow{\mu_6^*} (\tilde{V} \otimes_S \mathcal{N}^{(2)})^S[-h-2] \xleftarrow{\mu_5^*} \\ \cdots &\leftarrow A^S[-3h] \xleftarrow{\mu_{12}^*} \mathcal{N}^S[-2h-3] \xleftarrow{\mu_{11}^*} (V \otimes_S \mathcal{N})^S[-2h-3] \xleftarrow{\mu_{10}^*} (\tilde{V} \otimes_S \mathcal{N})^S[-2h-2] \xleftarrow{\mu_9^*} \end{aligned}$$

**Proposition 4.1** *We have  $\mu_i^* = \pm \mu'_{16-i}$ .*

*Proof:* (i)  $\mu_1^* = -\mu'_3$ : Let  $a \in V$  and  $x \in A^S$ . Then

$$\mu_1^*(x)(1 \otimes a \otimes 1) = x \circ \mu_1(1 \otimes a \otimes 1) = x \circ (a \otimes 1 - 1 \otimes a) = ax - xa.$$

So  $\mu_1^*(x)$  maps  $1 \otimes a \otimes 1 \mapsto [a, x]$ , giving  $\mu_1^*(x) = \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes [a, x] = -\mu'_3(x)$ . Similarly,  $\mu_5^*(x)$  maps  $1 \otimes a \otimes 1 \mapsto ax - x\beta(a)$ , giving  $\mu_5^*(x) = \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes (ax - x\beta(a)) = -\mu'_{11}(x)$ , and we also have  $\mu_9^*(x) = \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes (ax - x\beta^2(a)) = -\mu'_7(x)$ .

(ii)  $\mu_2^* = \mu'_2$ : Let  $a' \in V$  and for each  $a \in V$  let  $x_a$  be a homogeneous element in  $A$  such that  $\tilde{a} \otimes x_a \in (\tilde{V} \otimes A)^S$ . Then

$$\begin{aligned} \mu_2^*\left(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a\right)(1 \otimes \tilde{a}' \otimes 1) &= \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a \circ \mu_2(1 \otimes \tilde{a}' \otimes 1) \\ &= \sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a \circ \left( \sum_{b, b' \in \mathcal{G}_1} W_{a'bb'}(b \otimes b' \otimes 1 + 1 \otimes b \otimes b') \right) = \sum_{b, b' \in \mathcal{G}_1} W_{a'bb'}(bx_{b'} + x_b b'). \end{aligned}$$

So  $\mu_2^*(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a)$  maps  $1 \otimes \tilde{a}' \otimes 1 \mapsto \sum_{b, b' \in \mathcal{G}_1} W_{a'bb'}(bx_{b'} + x_b b')$ , giving  $\mu_2^*(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a) = \sum_{a, b, b' \in \mathcal{G}_1} W_{abb'}(a \otimes bx_{b'} + a \otimes x_b b') = \mu'_2(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a)$ . Similarly,  $\mu_6^*(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a) = \sum_{a, b, b' \in \mathcal{G}_1} W_{abb'}(a \otimes bx_{b'} + a \otimes x_b \beta(b')) = \mu'_{10}(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a)$  and  $\mu_{10}^*(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a) = \sum_{a, b, b' \in \mathcal{G}_1} W_{abb'}(a \otimes bx_{b'} + a \otimes x_b \beta^2(b')) = \mu'_6(\sum_{a \in \mathcal{G}_1} \tilde{a} \otimes x_a)$ .

(iii)  $\mu_3^* = -\mu'_1$ : For each  $a \in V$  let  $y_a$  be a homogeneous element in  $A$  such that  $a \otimes y_a \in (V \otimes A)^S$ . Then

$$\begin{aligned} \mu_3^*\left(\sum_{a \in \mathcal{G}_1} a \otimes y_a\right)(1 \otimes 1) &= \sum_{a \in \mathcal{G}_1} a \otimes y_a \circ \mu_3(1 \otimes 1) \\ &= \sum_{a \in \mathcal{G}_1} a \otimes y_a \circ \sum_{b \in \mathcal{G}_1} (b \otimes \tilde{b} \otimes 1 - 1 \otimes \tilde{b} \otimes b) = \sum_{b \in \mathcal{G}_1} (by_b - y_b b). \end{aligned}$$

So  $\mu_3^*(\sum_{a \in \mathcal{G}_1} a \otimes y_a)$  maps  $1 \otimes 1 \mapsto \sum_{b \in \mathcal{G}_1} [b, y_b]$ , giving  $\mu_3^*(\sum_{a \in \mathcal{G}_1} a \otimes y_a) = \sum_{a \in \mathcal{G}_1} [a, y_a] = -\mu'_1(\sum_{a \in \mathcal{G}_1} a \otimes y_a)$ . Similarly,  $\mu_7^*(\sum_{a \in \mathcal{G}_1} a \otimes y_a) = \sum_{a \in \mathcal{G}_1} (ay_a - y_a \beta(a)) = -\mu'_9(\sum_{a \in \mathcal{G}_1} a \otimes y_a)$  and  $\mu_{11}^*(\sum_{a \in \mathcal{G}_1} a \otimes y_a) = \sum_{a \in \mathcal{G}_1} (ay_a - y_a \beta^2(a)) = -\mu'_5(\sum_{a \in \mathcal{G}_1} a \otimes y_a)$ .

(iv)  $\mu_4^* = \mu'_{12}$ : Let  $x \in A^S$ . Then

$$\mu_4^*(x)(1 \otimes 1) = x \circ \mu_4(1 \otimes 1) = x \circ \sum_j w_j \otimes w_j^* = \sum_j w_j x w_j^*,$$

where  $\{w_j\}$  is a homogeneous basis for  $A$  and  $\{w_j^*\}$  is its corresponding dual basis. So  $\mu_4^*(x)$  maps  $1 \otimes 1 \mapsto \sum_j w_j x w_j^*$ , giving  $\mu_4^*(x) = \sum_j w_j x w_j^* = \mu_{12}'(x)$ . Similarly,  $\mu_8^*(x) = \sum_j w_j x \beta(w_j^*) = \mu_8'(x)$  and  $\mu_{12}^*(x) = \sum_j w_j x \beta^2(w_j^*) = \mu_4'(x)$ .  $\square$

Thus we see that we can identify, up to a shift in degree, each four-term portion of the cohomology complex (24) with a portion of the homology complex (11):

$$\begin{aligned} HH^i(A) &\cong HH_{3-i}(A)[-3], & i = 1, 2, \\ HH^i(A) &\cong HH_{15-i}(A)[-3h-3], & i = 3, \dots, 12, \\ HH^{12+i}(A) &\cong HH^i(A)[-3h], & i = 1, 2, \dots, \end{aligned}$$

and the self-duality of the homology complex (11) yields the relations

$$\begin{aligned} HH^i(A)^* &\cong HH^{7-i}(A), & i = 1, \dots, 6, \\ HH^i(A)^* &\cong HH^{19-i}(A), & i = 7, \dots, 11. \end{aligned}$$

## 4.2 The Hochschild cohomology of $A = A(\mathcal{G}, W)$

For  $HH^0(A) = \ker(\mu_1^*)/\text{Im}(\mu_0^*) = \ker(\mu_1^*)$ , we have  $HH^0(A) \cong HH_3(A)'[-3] \oplus L$ , where  $HH_3(A)' = \bigoplus_{j=3}^{h-1} HH_3(A)_j$  and  $L = \mathbb{C}\{u_{j\nu(j)} \mid \nu(j) = j\}$ .

Then we have the following results for the Hochschild cohomology of  $A$ :

**Theorem 4.2** *The Hochschild cohomology of  $A = A(\mathcal{A}^{(4)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(4)}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong X[-3], & HH^1(A) &\cong X[-3], \\ HH^6(A) &\cong X^*[-3], & HH^7(A) &\cong X^*[-3], \\ HH^{12}(A) &\cong X[-15], & HH^j(A) &= 0, \quad j = 2, \dots, 5, 8, \dots, 11, \end{aligned}$$

and  $HH^{12+i}(A) \cong HH^i(A)[-12]$  for  $i \geq 1$ , where the graded vector space  $X$  has Hilbert series  $H_X(t) = t^3$ .

**Theorem 4.3** *The Hochschild cohomology of  $A = A(\mathcal{A}^{(5)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(5)}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong X_2[-3], & HH^1(A) &\cong X_2[-3], \\ HH^2(A) &= 0, & HH^3(A) &= 0, \\ HH^4(A) &= 0, & HH^5(A) &= 0, \\ HH^6(A) &\cong X_2^*[-3], & HH^7(A) &\cong X_2^*[-3], \\ HH^8(A) &\cong X_3^*[-3], & HH^9(A) &\cong X_3^*[-3], \\ HH^{10}(A) &\cong X_3[-18], & HH^{11}(A) &\cong X_3[-18], \\ HH^{12}(A) &\cong X_2[-18], & HH^{12+i}(A) &\cong HH^i(A)[-15], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $X_2$  and  $X_3$  have Hilbert series  $H_{X_2}(t) = t^3$  and  $H_{X_3}(t) = t^6$  respectively.

**Theorem 4.4** *The Hochschild cohomology of  $A = A(\mathcal{A}^{(6)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(6)}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong X[-3] \oplus L, & HH^1(A) &\cong X[-3], \\ HH^2(A) &= 0, & HH^3(A) &\cong K^*[-3], \\ HH^4(A) &\cong K[-3], & HH^5(A) &= 0, \\ HH^6(A) &\cong X^*[-3], & HH^7(A) &\cong X^*[-3], \\ HH^8(A) &= 0, & HH^9(A) &= 0, \\ HH^{10}(A) &= 0, & HH^{11}(A) &= 0, \\ HH^{12}(A) &\cong X[-21], & HH^{12+i}(A) &\cong HH^i(A)[-18], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $L$ ,  $X$  and  $K$  have Hilbert series  $H_L(t) = t^3$ ,  $H_X(t) = t^3$  and  $H_K(t) = 2$  respectively.

**Theorem 4.5** *The Hochschild cohomology of  $A = A(\mathcal{A}^{(7)}, W)$ , where  $W$  is any cell system on  $\mathcal{A}^{(7)}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong X_2[-3], & HH^1(A) &\cong X_2[-3], \\ HH^2(A) &= 0, & HH^3(A) &\cong K^*[-3], \\ HH^4(A) &\cong K[-3], & HH^5(A) &= 0, \\ HH^6(A) &\cong X_2^*[-3], & HH^7(A) &\cong X_2^*[-3], \\ HH^8(A) &\cong X_3^*[-3], & HH^9(A) &\cong X_3^*[-3], \\ HH^{10}(A) &\cong X_3[-24], & HH^{11}(A) &\cong X_3[-24], \\ HH^{12}(A) &\cong X_2[-24], & HH^{12+i}(A) &\cong HH^i(A)[-21], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $X_2$ ,  $X_3$  and  $K$  have Hilbert series  $H_{X_2}(t) = t^3 + t^6$ ,  $H_{X_3}(t) = t^9$  and  $H_K(t) = 2$  respectively.

**Theorem 4.6** *The Hochschild cohomology of  $A = A(\mathcal{D}^{(6k)}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems given in [21], is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[6k-3] \oplus L, & HH^1(A) &\cong C^*[6k-3] \oplus X^*[6k-3], \\ HH^2(A) &\cong C[-3] \oplus X[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^{4+i}(A) &\cong HH^i(A)[-6k], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$ ,  $X$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{2k-2} 3t^{3j} + t^{6k-3}$ ,  $H_L(t) = (3k(2k-1)+3)t^{6k-3}$ ,  $H_X(t) = t^3 + \sum_{j=2}^{2k-2} 3t^{3j} + t^{6k-3}$  and  $H_K(t) = 6k(k-1)+2$  respectively, where for  $k = 1$ ,  $H_X(t) = 0$ .

**Theorem 4.7** *The Hochschild cohomology of  $A = A(\mathcal{D}^{(6k+3)}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems given in [21], is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[6k] \oplus L, & HH^1(A) &\cong C^*[6k] \oplus X^*[6k], \\ HH^2(A) &\cong C[-3] \oplus X[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^{4+i}(A) &\cong HH^i(A)[-6k-3], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$ ,  $X$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{2k-1} 3t^{3j} + t^{6k}$ ,  $H_L(t) = (3k(2k+1)+3)t^{6k}$ ,  $H_X(t) = t^3 + \sum_{j=2}^{2k-1} 3t^{3j} + t^{6k}$  and  $H_K(t) = 6k^2$  respectively.



**Theorem 4.8** *The Hochschild cohomology of  $A = A(\mathcal{A}^{(n)*}, W)$ ,  $n \geq 4$ , where  $W$  is any cell system on  $\mathcal{A}^{(n)*}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[n-3] \oplus L, & HH^1(A) &\cong C^*[n-3], \\ HH^2(A) &\cong C[-3], & HH^3(A) &\cong C[-3], \\ HH^4(A) &\cong C^*[-3], & HH^{4+i}(A) &\cong HH^i(A)[-n], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$  have Hilbert series  $H_C(t) = \sum_{j=1}^{n-3} \lfloor (n-j-1)/2 \rfloor t^j$  and  $H_L(t) = \lfloor (n-1)/2 \rfloor t^{n-3}$ .

**Theorem 4.9** *The Hochschild cohomology of  $A = A(\mathcal{D}^{(6k)*}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems given in [21], is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[6k-3] \oplus L, & HH^1(A) &\cong C^*[6k-3], \\ HH^2(A) &\cong C[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^{4+i}(A) &\cong HH^i(A)[-6k], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-1)/3 \rfloor} (m - \lfloor 3j/2 \rfloor) t^{3j}$ ,  $H_L(t) = (9k-3)t^{6k-3}$  and  $H_K(t) = 6k-4$  respectively.

**Theorem 4.10** *The Hochschild cohomology of  $A = A(\mathcal{D}^{(6k+3)*}, W)$ ,  $k \geq 1$ , where  $W$  is equivalent to one of the cell systems given in [21], is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[6k] \oplus L, & HH^1(A) &\cong C^*[6k], \\ HH^2(A) &\cong C[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^{4+i}(A) &\cong HH^i(A)[-6k-3], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$ ,  $K$  have Hilbert series  $H_C(t) = \sum_{j=1}^{\lfloor (2m-2)/3 \rfloor} (m - \lfloor (3j+1)/2 \rfloor) t^{3j}$ ,  $H_L(t) = (9k+3)t^{6k}$  and  $H_K(t) = 6k$  respectively.

**Theorem 4.11** *The Hochschild cohomology of  $A = A(\mathcal{E}^{(8)}, W)$ , where  $W$  is any cell system on  $\mathcal{E}^{(8)}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong X_2[-3], & HH^1(A) &\cong X_2[-3], \\ HH^2(A) &\cong C[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^5(A) &\cong C^*[-3], \\ HH^6(A) &\cong X_2^*[-3], & HH^7(A) &\cong X_2^*[-3], \\ HH^8(A) &\cong X_3^*[-3], & HH^9(A) &\cong X_3^*[-3], \\ HH^{10}(A) &\cong X_3[-27], & HH^{11}(A) &\cong X_3[-27], \\ HH^{12}(A) &\cong X_2[-27], & HH^{12+i}(A) &\cong HH^i(A)[-24], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $X_2$ ,  $X_3$  and  $K$  have Hilbert series  $H_C(t) = t^3$ ,  $H_{X_2}(t) = t^3 + t^6$ ,  $H_{X_3}(t) = 2t^9$  and  $H_K(t) = 2$  respectively.

**Theorem 4.12** *The Hochschild cohomology of  $A = A(\mathcal{E}^{(8)*}, W)$ , where  $W$  is any cell system on  $\mathcal{E}^{(8)*}$ , is given by*

$$\begin{aligned} HH^0(A) &\cong C^*[5] \oplus L, & HH^1(A) &\cong C^*[5], \\ HH^2(A) &\cong C[-3], & HH^3(A) &\cong C[-3] \oplus K^*[-3], \\ HH^4(A) &\cong C^*[-3] \oplus K[-3], & HH^{4+i}(A) &\cong HH^i(A)[-8], \quad i \geq 1, \end{aligned}$$

where the graded vector spaces  $C$ ,  $L$ ,  $K$  have Hilbert series  $H_C(t) = 2t + t^2 + t^3 + t^5$ ,  $H_L(t) = 4t^5$  and  $H_K(t) = 2$  respectively.

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